Fall 2014 GEOL0350–GeoMath Midterm Exam

1 Critique

Late at night, a fellow student comes to you in the lab with a cocksure smile. "You know," he says, "I've solved the turbulence problem. Einstein and Feynman were stumped, but I've solved it by looking at my simulations. In these simulations, the velocity ($\mathbf{v}$) of a turbulent flow in meters per second is equal to the frequency (f) in cycles per minute. Therefore, I can tell you everything you want to know about turbulence." This "friend" has problems. The questions below will determine if you can help.

1.1 Units

What is wrong with the units in the theory? How would you convert the frequency units to fix the issue?

In the velocity, the time units are seconds. In the frequency they are minutes. Therefore, you should convert the frequency to seconds first, by multiplying by $\frac{\mathbf{1minute}}{60\,\mathrm{s}} = 1$.

1.2 Dimensions

What is wrong with the dimensions in the theory? If you ask for the viscosity used in the simulation (ν in m^2/s), can you find a dimensionless parameter? As there is only one, what must be true of its value (Here your friend has kind of the right idea...).

The dimensions of velocity are length per time, so they cannot be equated to a frequency in 1/T. However, with a viscosity you can form $\frac{\nu f}{v^2}$ which is dimensionless. As there is only one dimensionless quantity, it must be constant. Thus, $v^2 \propto \nu f$ is a more reasonable theory.

1.3 Vector

What is wrong with equating velocity to frequency even after you figure out the units and the dimensions? What quantity can you form from the velocity that would be more meaningful?

A velocity is a vector while frequency is a scalar. Thus, you would want to equate $\mathbf{v} \cdot \mathbf{v} \propto \nu f$ so that you have a scalar equal to a scalar, i.e., the magnitude of the velocity (speed) can be related to the frequency.

2 Error Detection

Yesterday, Maria Zuber from MIT gave the department colloquium on results from a measurement of the gravity field of the moon. The measurements were of unprecedented accuracy, but there were a few unexpected signals in the data that needed to be removed. That talk inspired this question.

2.1 Force of Gravity

As we know from our homework, gravity is a force that can be generated from a gravitational potential $\mathbf{F}_g = -\nabla\phi_g$. Answer the following with this fact in mind.

2.1.1 Divergence

Express $\nabla \cdot \mathbf{F}_g$ in terms of the gravitational potential ϕ_g.

$$\nabla \cdot \mathbf{F}_g = -\nabla \cdot \nabla\phi_g = -\nabla^2\phi_g.$$

2.1.2 Curl

Express $\nabla \times \mathbf{F}_g$ in terms of the gravitational potential ϕ_g.

$$\nabla \times \mathbf{F}_g = 0.$$

2.2 Errors in Gravity Measurement

The measured force $\mathbf{F}_m$ can be decomposed, as any vector can be decomposed ($\mathbf{F}_m = \mathbf{F}_d + \mathbf{F}_r$), into a part that is generated by a scalar potential ($\mathbf{F}_d = -\nabla\phi_m$) and a part that is generated by a vector potential ($\mathbf{F}_r = \nabla \times \psi_m$). If the measurement of the gravitational force field is decomposed in this way, **quantify the error in measured gravitational force implied by the presence of $\mathbf{F}_r$ in magnitude and direction using only ψ_m and ϕ_m.** (Hint: Use $|\Delta\mathbf{F}|/|\mathbf{F}_{\text{true}}|$ for magnitude error and cosine of the angle between the possibly true and the total measured force to quantify the direction error).

We associate $\mathbf{F}_d$ with $\mathbf{F}_g$, as gravity is a conservative force so all non-conservative forces, $\mathbf{F}_r$, must be erroneous.

$$E_m = \frac{|\mathbf{F}_r|}{|\mathbf{F}_d|} = \sqrt{\frac{\nabla \times \psi_m \cdot \nabla \times \psi_m}{\nabla\phi_m \cdot \nabla\phi_m}},$$

$$\cos\theta = \frac{F_m \cdot \mathbf{F}_d}{|F_m||F_d|} = \frac{(\mathbf{F}_d + \mathbf{F}_r) \cdot \mathbf{F}_d}{|\mathbf{F}_d + \mathbf{F}_r||\mathbf{F}_r|} = \frac{(-\nabla\phi_m + \nabla \times \psi_m) \cdot (-\nabla\phi_m)}{|-\nabla\phi_m + \nabla \times \psi_m||\nabla \times \psi_m|}$$

3 Getting Seri-us

The Taylor series and the Fourier series are methods to determine the coefficients in series expansions of a function $f(x)$. They apply in approximating the function near $x = a$ and over a given interval where the function is periodic. Suppose we have a function that is periodic when $-\pi \leq x - a \leq \pi$, then we can write the Taylor and Fourier series–for the *same function* $f(x)$ as,

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots,$$
$$f(x) = d_0 + d_1 e^{i(x-a)} + d_2 e^{2i(x-a)} + d_3 e^{3i(x-a)} + \dots$$

Use these forms to answer the following questions. Hint:

$$e^{ni(x-a)} = 1 + in(x-a) - \frac{n^2}{2}(x-a)^2 - \frac{in^3}{6}(x-a)^3 + \dots + \frac{(in)^m}{m!}(x-a)^m + \dots$$

3.1 Constant

Is there a simple relationship between c_0 and d_0 very near $x = a$ for all $f(x)$?

No. Setting $x = a$ yields $c_0 = d_0 + d_1 + d_2 + d_3 + \dots$

3.2 Variations

Does a specific term in the Fourier series, say $d_2 e^{2i(x-a)}$ always correspond to a specific term in the Taylor series, regardless of what $f(x)$ is?

No. Every term in the Fourier series will contribute to every term in the Taylor series, since each has a part of it that goes as $(x-a)$ to any given power, as seen clearly in the hint.

3.3 Equivalence?

Assuming both series converge, is there a unique (though perhaps complicated) relationship among all of the c_n and all of the d_n?

Yes. There is a unique Taylor series and a unique Fourier series for a given function $f(x)$, and either can be used to derive the other, so there is a unique relationship among all of the coefficients of each.

3.4 Specifically

If $f(x) = e^{i(x-a)}$, what are the values of all nonzero d_n and c_n?

$d_1 = 1$, all other d_n are zero. c_n are the coefficients given in the hint, $c_m = \frac{(i)^m}{m!}$.

4 Linear Integrals and Derivatives

Use the following fact to rapidly calculate the results below.

$$\nabla \frac{r^2}{2} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} = \mathbf{r}$$

4.1 Q1

$$\nabla \times \left\{ \nabla \times (12y\hat{\mathbf{i}} - 12x\hat{\mathbf{j}}) + 4\left(\nabla \times \left([3x - 3y]\hat{\mathbf{i}} + [3x + 3y]\hat{\mathbf{j}} + 3z\hat{\mathbf{k}}, \right) \right) \right\} =$$

$$= \nabla \times \left\{ \nabla \times (12y\hat{\mathbf{i}} - 12x\hat{\mathbf{j}}) + \left(\nabla \times \left([12x - 12y]\hat{\mathbf{i}} + [12x + 12y]\hat{\mathbf{j}} + 12z\hat{\mathbf{k}}, \right) \right) \right\},$$

$$= \nabla \times \left\{ \left(\nabla \times \left(12x\hat{\mathbf{i}} + 12y\hat{\mathbf{j}} + 12z\hat{\mathbf{k}}, \right) \right) \right\},$$

$$= 12\nabla \times \left\{ \left(\nabla \times \left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}, \right) \right) \right\}.$$

The answer is 12 times $\nabla \times \left[\nabla \times \nabla \frac{r^2}{2} \right]$, and the curl of any gradient is zero, so 0.

4.2 Q2

$$\int_{\pi}^{X} 2x \, \mathrm{d}x + \int_{\pi}^{Y} 2y \, \mathrm{d}y + \int_{\pi}^{Z} 2z \, \mathrm{d}z =$$

This is the line integral of twice the gradient given from (π, π, π) to (X, Y, Z), so $= X^2 + Y^2 + Z^2 - 3\pi^2$.

4.3 Q3

Integrating around any closed loop:

$$\oint \mathbf{r} \cdot \mathrm{d}\mathbf{l} =$$

$= \iint \nabla \times \nabla \frac{r^2}{2} \, \mathrm{d}A = 0$, alternatively it equals the value of the potential at the start point minus the end point $= \frac{r^2}{2} - \frac{r^2}{2}$.