Fall 2015 GEOL0350–GeoMath Midterm Exam

1 Seismic Waves

The seismic wave equation can be written

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = \alpha^2 \nabla(\nabla \cdot \mathbf{u}) - \beta^2 \nabla \times (\nabla \times \mathbf{u}). \tag{1}$$

The vector $\mathbf{u}$ is a function of space and time, and it measures the displacement of every point in a solid from its original location. The two parameters α and β are constants.

1.1 Fields

What kind of a mathematical object is $\mathbf{u}$? $\mathbf{u}$ is a vector field.

1.2 Dimensions

What are the dimensions of α and β? They both have the dimensions of velocity, L/T.

1.3 Split It Up

Suppose we write the displacement as a combination of a scalar potential ϕ and a vector streamfunction $\boldsymbol{\psi}$ (a Helmholtz decomposition), that is $\mathbf{u} = \nabla\phi + \nabla \times \boldsymbol{\psi}$. Plug this assumed form into the seismic wave equation, and collect all of the terms that depend only on ϕ on the left hand side and those that depend only on $\boldsymbol{\psi}$ on the right hand side. Simplify where you can easily.

$$\begin{aligned}
\frac{\partial^2 \nabla\phi + \nabla\times\boldsymbol{\psi}}{\partial t^2} = \frac{\partial^2 \nabla\phi}{\partial t^2} + \frac{\partial^2 \nabla\times\boldsymbol{\psi}}{\partial t^2} &= \alpha^2 \nabla(\nabla \cdot [\nabla\phi + \cancelto{0}{\nabla \times \boldsymbol{\psi}}]) - \beta^2 \nabla \times (\nabla \times [\cancelto{0}{\nabla\phi} + \nabla \times \boldsymbol{\psi}]), \\
\frac{\partial^2 \nabla\phi}{\partial t^2} - \alpha^2 \nabla^2 \nabla\phi &= -\frac{\partial^2 \nabla\times\boldsymbol{\psi}}{\partial t^2} - \beta^2 \nabla \times (\nabla \times [\nabla \times \boldsymbol{\psi}]) \\
&= -\frac{\partial^2 \nabla\times\boldsymbol{\psi}}{\partial t^2} - \beta^2 \nabla\cancelto{0}{(\nabla \cdot [\nabla \times \boldsymbol{\psi}])} + \beta^2 \nabla^2 (\nabla \times \boldsymbol{\psi}),
\end{aligned}$$

1.4 Solution

The seismic P-wave is associated with ϕ and the S-wave is associated with $\boldsymbol{\psi}$. Show that $\phi = f(x - \alpha t)$, for any function f makes the left hand side of the equation from the last part equal zero.

$$\frac{\partial^2 \nabla f(x-\alpha t)}{\partial t^2} - \alpha^2 \nabla^2 \nabla f(x - \alpha t) = \alpha^2 \hat{\mathbf{x}} f'''[x - \alpha t] - \alpha^2 \hat{\mathbf{x}} f'''[x - \alpha t] = 0.$$

2 Linear and Nonlinear

We have talked a lot about linear and nonlinear functions and equations. This question explores these ideas.

2.1 Equations for a Line

Show that $y = mx$ is linear, by considering $x = 2x_0$ and $x = x_1 + x_2$, with m being constant.

$$y = mx = m(x_1 + x_2) = mx_1 + mx_2 = y_1 + y_2, \qquad y = m2x_0 = 2mx_0 = 2y_0.$$

2.2 Parabolic

Use the same approach to show that $y = mx^2$ is not linear.

$$y = mx^2 = m(x_1 + x_2)^2 = mx_1^2 + mx_2^2 + 2mx_1x_2 \neq mx_1^2 + mx_2^2 = y_1 + y_2, \qquad y = m(2x_0)^2 = 4mx_0^2 = 4y_0.$$

2.3 Differential

Use the same approach to show that $x = \frac{\partial x}{\partial t}$ is linear.

$$x = \frac{\partial x}{\partial t} = \frac{\partial x_1 + x_2}{\partial t} = \frac{\partial x_1}{\partial t} + \frac{\partial x_2}{\partial t} = x_1 + x_2, \qquad x = 2x_0 = \frac{\partial 2x_0}{\partial t} = 2\frac{\partial x_0}{\partial t} = 2x_0.$$

2.4 Linear Equations, Nonlinear Solutions?

In class, we saw that the solution to $x = \frac{\partial x}{\partial t}$ was $x = Ce^t$. Is this equation linear? Is this solution a line? What does this tell you about linear equations? Equation is linear, solution is not a line. Thus, linear equations can be solved by lines sometimes, but other times other functions. Just because the solution is not a line doesn't mean the equation is not linear.

3 Taylor rhymes with Baylor

Suppose the function $h(x)$ plotted in the figure is found by measuring topography along the x direction, and in particular consider fitting it with a series expansion near the point marked A. Sea level is $h = 0$, so we are particularly interested in $h(x) = 0$, which indicates the location of coastlines. The function is not known, but we consider the possibility of approximating a Taylor series expansion to it.

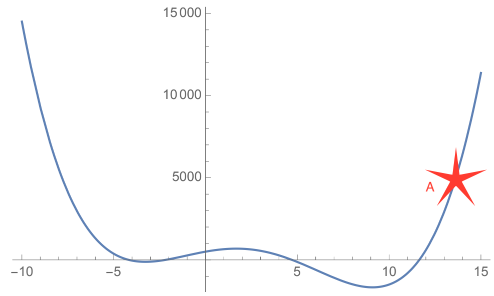

Figure 1: A function to be fit by Taylor expansion near point A at the star.

3.1 Counting

How many coastlines are there, that is how many solutions to $h(x) = 0$ are there? 4

3.2 Constant

If the function is fit with a Taylor series and truncated at the first (constant) term. If the truncated approximation is denoted $\tilde{h}_0(x)$, then how many solutions are there to $\tilde{h}_0(x) = 0$? None

3.3 Variations

If the Taylor series is instead truncated after two terms ($\tilde{h}_1(x)$) then how many solutions are there to $\tilde{h}_1(x) = 0$, and what is the approximate value of the solution x? One, near $x = 12$.

3.4 How many?

What is the minimum number of terms in the Taylor series that must be retained to approximate all of the coastlines in the real function $h(x)$? Why? Since there are 4 roots to the equation $h(x) = 0$, $\tilde{h}(x)$ must be at least a fourth-order polynomial to have this many roots, which has 5 terms of the Taylor series retained.

3.5 Extremes

Consider the function at large magnitudes of positive and negative x. If the truncated Taylor series matches this behavior at large $|x|$, predict the sign of the coefficient in the largest power of x and whether the power is even or odd. Positive and even power. Negative would tend toward $-\infty$, and even power since the function increases on both sides of A or 0.

4 Impossible!

Indicate why the following equations cannot be true.

4.1 Q1

$$\sin(kx) \approx a + bx^2 + cx^4 + \dots$$

Sine is odd and the RHS is even.

4.2 Q2

$$f(\sin^2\theta + \cos^2\theta) = \theta$$

The function on the left must be constant in theta, since its argument is independent of θ, while the other side depends on θ.

4.3 Q3

$$\oint \mathbf{v} \cdot \mathrm{d}\mathbf{l} = \mathbf{v}$$

At first, it seems like the LHS would be zero, but it might not be if $\nabla \times v \neq 0$. However, the LHS is a scalar and the RHS is a vector!

4.4 Q4

$$\oiint (\nabla \times \mathbf{v}) \cdot \hat{\mathbf{n}}\, \mathrm{d}A = 1$$

By the divergence theorem, $\oiint (\nabla \times \mathbf{v}) \cdot \hat{\mathbf{n}}\, \mathrm{d}A = \iiint \nabla \cdot (\nabla \times \mathbf{v})\, \mathrm{d}V = 0$ so the LHS is zero.

4.5 Q5

Speed always equals speed squared.

Two problems. 1) Units/dimensions don't match. 2) $s = s^2$ only if $s = 1$, not always.