1 Centripetal

Problem 6.3.17 Boas (2006). 6.3.17: Expand the triple product $\mathbf{a} = \omega \times \omega \times \mathbf{r}$ given in the discussion of this figure. If $\mathbf{r}$ is perpendicular to $\omega$ (Problem 16), show that $a = -\omega^2 \mathbf{r}$, and so find the elementary result that the acceleration is toward the center of the circle and of magnitude $v^2/r$.

Applications of the Triple Vector Product  In Figure 3.8 (compare Figure 2.6), suppose the particle $m$ is at rest on a rotating rigid body (for example, the earth). Then the angular momentum $L$ of $m$ about point $O$ is defined by the equation $L = \mathbf{r} \times (m \mathbf{v}) = m \mathbf{r} \times \mathbf{v}$. In the discussion of Figure 2.6, we showed that $\mathbf{v} = \omega \times \mathbf{r}$. Thus, $L = m \mathbf{r} \times (\omega \times \mathbf{r})$. See Problem 16 and also Chapter 10, Section 4.

As another example, it is shown in mechanics that the centripetal acceleration of $m$ in Figure 3.8 is $\mathbf{a} = \omega \times (\omega \times \mathbf{r})$. See Problem 17.

2 Direction of Decrease

Problem 6.6.2 of Boas (2006). Starting from the point (1,1), in what direction does the function $\phi = x^2 - y^2 + 2xy$ decrease most rapidly?

3 Calculate Div, Grad, Curl

Problem 6.7.1 Boas (2006). Compute the divergence and curl of $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

4 A Triple Product

Problem 6.7.18 Boas (2006). For $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, evaluate $\nabla \times (\mathbf{k} \times \mathbf{r})$
5 Check for Conservative—Attraction to the origin

Problem 6.8.10 of Boas (2006). This problem was selected because it is very similar to Hooke’s law, which is a fundamental of solid mechanics and gravitational attraction. Verify that the following force field is conservative. Then find, for each, a scalar potential \( \phi \) such that \( F = -\nabla \phi \).

\[
\begin{align*}
F &= -kr, \\
r &= x\hat{i} + y\hat{j} + z\hat{k}, \\
k &= \text{constant.}
\end{align*}
\]

6 Gradient of \( r \)

The combination of gravitation and the centrifugal force from the earth’s rotation is a conservative force that can be expressed using the geopotential \( \phi = mgz \), where \( z \) is distance from the surface and \( m \) and \( g \) are the constant mass and acceleration due to gravity. A motion that results in a change in geopotential indicates the possibility that energy can be extracted from the motion. Formulate a closed line integral (using Stokes theorem) for a route to school that proves that “in my day, we had to go to school uphill both ways!” cannot require a net expenditure of energy. Show that if there is a nonconservative force (e.g., viscosity of air, rusty bike wheels, etc.) there may be a nonzero expenditure of energy in the round trip.

7 Stokes Theorem

a) Calculate the curl of the vector \( \mathbf{v} = (-y, x, 0) \). b) Take the area integral of \( (\nabla \times \mathbf{v}) \cdot \hat{n} \) over the surface of a disc bounded by \( 1 = x^2 + y^2 \). c) Take the area integral of \( (\nabla \times \mathbf{v}) \cdot \hat{n} \) over the surface of the half-sphere bounded by \( 1 = x^2 + y^2 + z^2 \) where \( z \geq 0 \). d) Use Stokes’ theorem to find a line integral equal to both b) and c).

8 Divergence Theorem

a) Calculate the divergence of the vector \( \mathbf{v} = (-y, x, 1) \). b) What is the area integral of \( \iint \mathbf{v} \cdot \hat{n} \, dS \), over any given closed area? c) Can you say anything about its integral over an area that is not closed?

References