Assignment 1 for GEOL 1820: Geophysical Fluid Dynamics, Waves and Mean Flows Edition Due Sept. 23, 2016

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October 3, 2016

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Getting Help!

I am usually available by email. You can make an appointment other times. Just check my calendar at http://fox-kemper.com/contact and suggest a time that works for you.

1 Problem 1

Revisit the derivations of mass conservation in flux form (Bühler, 2014, Section 1.1.1) and material derivative form (Bühler, 2014, Section 1.1.4 equation (1.15)).

1.1 Flux Form

Derive a flux form differential equation for the conservation of salt, expressed in terms of salinity (a mass fraction of seawater density, e.g., in grams per kilogram), assuming that the only way that salt moves is by fluid advection and that no chemical reactions create or destroy salt. Do not assume that the fluid is incompressible.

This problem begins by assuming that the mass of salt is conserved. Thus, for any volume $\mathcal{D}$,

$$M_s = \int_{\mathcal{D}} \rho S \, dV, \tag{1}$$

$$0 = \frac{dM_s}{dt} = \frac{d}{dt}\int_{\mathcal{D}} \rho S \, dV = \int_{\mathcal{D}} \frac{\partial(\rho S)}{\partial t} dV + \oint_{\partial\mathcal{D}} \rho S \mathbf{u} \cdot d\mathbf{A} = \int_{\mathcal{D}} \left(\frac{\partial(\rho S)}{\partial t} + \nabla \cdot (\rho S \mathbf{u}) \right) dV. \tag{2}$$

Since the volume chosen is arbitrary, the integrand must vanish everywhere for the integral to vanish, thus

$$\frac{\partial(\rho S)}{\partial t} + \nabla \cdot (\rho S \mathbf{u}) = 0. \tag{3}$$

The continuity equation can be exploited to write for salinity alone:

$$\rho \frac{\partial S}{\partial t} + \rho \mathbf{u} \cdot \nabla S = -S \frac{\partial \rho}{\partial t} - S \nabla \cdot (\rho \mathbf{u}) = 0, \tag{4}$$

$$\frac{DS}{Dt} = 0. \tag{5}$$

1.2 Material Derivative Form

Derive a material derivative form equation for the conservation of salt working from Section 1.1.4 equation (1.15), again expressed in terms of salinity (a mass fraction of seawater density, e.g., in grams per kilogram), assuming that the only way that salt moves is by fluid advection and that no chemical reactions create or destroy salt.

If we use $\phi = S$ in equation (1.15), expressing the rate of change of a material volume,

$$0 = \frac{d}{dt} \int_{\mathcal{D}_{\mathcal{M}}} \rho S \, dV = \int_{\mathcal{D}_{\mathcal{M}}} \frac{D}{Dt} (\rho S \, dV) = \int_{\mathcal{D}_{\mathcal{M}}} \rho \frac{DS}{Dt} \, dV. \tag{6}$$

Since the material volume chosen is arbitrary, the integrand must vanish everywhere for the integral to vanish, thus

$$0 = \rho \frac{DS}{Dt}, \tag{7}$$

$$0 = \frac{DS}{Dt}. \tag{8}$$

1.3 Diffusion

Salt may also move through water by diffusion (i.e., molecular transport down any large-scale gradients), which does not require advection. Does this affect your salt budget equations? Why do you think diffusion of fluid mass is generally ignored?

Yes, it does affect the budget equations significantly. For the flux form budget, the diffusive flux of salt through the surface must be added in addition to the advective flux. For the material volume form, the rate of change of salt cannot be taken to be zero, as indeed it may decrease or increase within the material volume by diffusion.

Diffusion of mass in the advective equation is typically ignored because of the continuum hypothesis. When the molecular diffusion of mass is important is when it there are regions where the density jumps suddenly (e.g., boiling of water when exposed to a vacuum). Typically, these processes are represented through boundary conditions rather than diffusion, because then there are no issues with the continuum hypothesis.

2 Problem 2: Shallow water waves with rotation

Problem 1 in section 2.3 of (Bühler, 2014).

The rotating SWE and their linearized form about a state of rest with constant depth ($h = h' + H, \mathbf{u} = \mathbf{u}' + 0$) are:

$$\partial_t h + \nabla \cdot h\mathbf{u} = 0, \qquad \frac{D\mathbf{u}}{Dt} + \mathbf{f} \times \mathbf{u} + g\nabla h = 0. \tag{9}$$

$$\partial_t h' + H\nabla \cdot \mathbf{u}' + \cancel{\nabla \cdot \mathbf{h}'0} + \cancel{\nabla \cdot \mathbf{h}'u'} = 0, \qquad \partial_t \mathbf{u}' + \cancel{\mathbf{u}' \cdot \nabla \mathbf{u}'} + \mathbf{f} \times \mathbf{u}' + g\nabla h' + \cancel{g\nabla H} = 0. \tag{10}$$

$$\partial_t h' + H\nabla \cdot \mathbf{u}' = 0, \qquad \partial_t \mathbf{u}' + \mathbf{f} \times \mathbf{u}' + g\nabla h' = 0. \tag{11}$$

If we assume a plane wave form, these equations become

$$\begin{bmatrix} h' \\ u' \\ v' \end{bmatrix} = \begin{bmatrix} \hat{h} \\ \hat{u} \\ \hat{v} \end{bmatrix} e^{i(kx+ly-\omega t-\alpha)}, \qquad \begin{bmatrix} -i\omega & Hik & Hil \\ gik & -i\omega & -f \\ gil & f & -i\omega \end{bmatrix} \begin{bmatrix} \hat{h} \\ \hat{u} \\ \hat{v} \end{bmatrix} = 0 \tag{12}$$

If a nontrivial solution is to exist, the determinant of this matrix must vanish (i.e., the exception to Cramer's rule),

$$i\omega^3 + gHflk - gHflk - i\omega gHl^2 - i\omega f^2 - i\omega gHk^2 = 0, \tag{13}$$

$$\omega^3 - \omega gHl^2 - \omega f^2 - \omega gHk^2 = 0, \tag{14}$$

$$\omega(\omega^2 - f^2 - gH(k^2 + l^2)) = 0. \tag{15}$$

The PV, and its linearized form are

$$q = \frac{\nabla \times \mathbf{u} + \mathbf{f}}{h} = \frac{\nabla \times \mathbf{u}'}{H + h'} + \frac{\mathbf{f}}{H + h'}\frac{H - h'}{H - h'} = \frac{\nabla \times \mathbf{u}'}{H} + O(u'h') + \frac{\mathbf{f}H}{H^2 - h'^2} - \frac{\mathbf{f}h'}{H^2 - h'^2}, \tag{16}$$

$$= \frac{\nabla \times \mathbf{u}'}{H} - \frac{\mathbf{f}h'}{H^2} + \frac{\mathbf{f}}{H} + O(u'h', h'^2) = \frac{1}{H}(\partial_x v' - \partial_y u' - \frac{fh'}{H}) + \frac{\mathbf{f}}{H} + O(u'h', h'^2). \tag{17}$$

Thus, if we drop the overall constant multiplier H and the constant and higher order terms,

$$q' = \partial_x v' - \partial_y u' - \frac{fh'}{H}. \tag{18}$$

If we assume a (balanced) vortical streamfunction based on the steady solution to $\partial_t \mathbf{u}' + \mathbf{f} \times \mathbf{u}' + g\nabla h' = 0$, $\mathbf{f} \times \mathbf{u}' + g\nabla h' = 0$, then

$$\begin{bmatrix} h'_b \\ u'_b \\ v'_b \end{bmatrix} = \begin{bmatrix} f\psi'/g \\ -\partial_y \psi' \\ \partial_x \psi' \end{bmatrix}, \qquad q' = \nabla^2 \psi' - \frac{f^2}{gH}\psi' \tag{19}$$

The wave modes have zero PV, but as they have $h'_w \neq 0$, then they must also have relative vorticity as $q'_w = 0$ implies $\partial_x v'_w - \partial_y u'_w = \frac{f}{H} h'_w$.

3 Problem 3: Adiabatic invariance vs. the pit and the pendulum

Problem 3 in section 2.3 of (Bühler, 2014).

We consider a pendulum, for which the tangential force balance is

$$\partial_t \text{momentum} = \partial_t[m\partial_t(l(t)\phi)] = -mg\sin\phi. \tag{20}$$

If we linearize, and assume the rate of change of $l(t)$ is small compared to the typical (constant) pendulum frequency Ω_0, then we can arrive at a WKB solution.

$$\partial_t^2 \left(\frac{1}{\Omega^2(t)} \phi \right) = -\phi, \tag{21}$$

$$\phi = \phi_0(t) e^{i(\Omega_0 s(t) - \alpha)}, \Omega_0^2 \sim \Omega^2 = g/l. \tag{22}$$

Plugging in, we find

$$1 - \frac{\Omega_0^2 \dot{s}(t)^2}{\Omega^2(t)} + \frac{2i\Omega_0 \dot{s}(t) \dot{\phi}_0(t)}{\phi_0(t)\Omega^2(t)} - \frac{4i\Omega_0 \dot{s}(t) \dot{\Omega}(t)}{\Omega^3(t)} - \frac{i\Omega_0 \ddot{s}(t)}{\Omega^2(t)} + O(1) \tag{23}$$

The leading order equation (by assuming Ω_0 is large compared to time derivatives of the other variables) is the eikonal equation, which can be solved by integration up to a constant that can be absorbed into α. Here I use $\dot{}$ to indicate a time derivative.

$$\frac{\Omega(t)}{\Omega_0} = \pm \dot{s}(t), s(t) = \pm \int^t \frac{\Omega(t)}{\Omega_0} dt \tag{24}$$

The next order equation is

$$\frac{2\Omega_0 \dot{s}(t) \dot{\phi}_0(t)}{\phi_0(t)\Omega(t)^2} - \frac{4\Omega_0 \dot{s}(t) \dot{\Omega}(t)}{\Omega(t)^3} + \frac{\Omega_0 \ddot{s}(t)}{\Omega(t)^2} = 0, \tag{25}$$

$$\frac{2\dot{\phi}_0(t)}{\phi_0(t)\Omega(t)} - \frac{4\dot{\Omega}(t)}{\Omega(t)^2} + \frac{\dot{\Omega}(t)}{\Omega(t)^2} = 0, \tag{26}$$

$$\frac{\dot{\phi}_0(t)}{\phi_0(t)} = \frac{3\dot{\Omega}(t)}{2\Omega(t)}, \tag{27}$$

$$\phi_0(t) = \Omega(t)^{3/2}. \tag{28}$$

Thus,

$$\phi \approx c_+ \Omega(t)^{3/2} e^{i\left(\int^t \Omega(t) dt - \alpha\right)} + c_- \Omega(t)^{3/2} e^{-i\left(\int^t \Omega(t) dt - \alpha\right)}. \tag{29}$$

If we now consider the phase-averaged energy,

$$\bar{E} = \frac{1}{2} l(t)^2 \overline{\dot{\phi}^2} + \frac{1}{2} \Omega(t)^2 l(t)^2 \overline{\phi^2}, \tag{30}$$

$$\propto \left[\Omega(t)^5 l(t)^2 + \Omega(t)^5 l(t)^2 \right] = \frac{g^{5/2}}{l(t)^{5/2}} l(t)^2 \propto l^{-1/2}. \tag{31}$$

And if we consider the maximum of $|\phi|$ over an oscillation the the maximum of $|l(t)\dot{\phi}(t)|$,

$$\max |\phi| \propto \Omega(t)^{3/2} \propto l^{-3/4}, \tag{32}$$

$$\max |l(t)\dot{\phi}(t)| \propto l(t)\Omega(t)^{5/2} \propto l^{-1/4}. \tag{33}$$

So, to anticipate the Poe story, the phase-averaged energy is expected to *decrease*, the maximum angle is expected to decrease even faster, and the maximum speed at the bottom is expected to decrease slowly. In other words, $\bar{E}/\Omega(t)$ is expected to be constant, not $\bar{E}$. As the pendulum length increases, the frequency and energy decrease.

In Poe, the pendulum begins with "Its sweep was brief, and of course slow", but later our hero finds "The sweep of the pendulum had increased in extent by nearly a yard. As a natural consequence, its velocity was also much greater. But what mainly disturbed me was the idea that had perceptibly descended." Thus, Poe has it backward–unless there is a hidden motor driving the pendulum as it descends!

4 Problem 4: Baroclinic Vorticity

Consider (1.45) of section 1.4.2 of (Bühler, 2014). Draw three pictures of isolines of pressure and density each of which has uniform gradient (although possibly in different directions): 1) one that does not produce vorticity, 2) one that produces clockwise vorticity, and 3) one that produces counter-clockwise vorticity. It is easiest to assume that the pressure isolines (isobars) are nearly horizontal.

Following the vorticity equation,

$$\frac{D}{Dt}\left(\frac{\nabla \times \mathbf{u}}{\rho}\right) - \left(\frac{\nabla \times \mathbf{u}}{\rho}\right) \cdot (\nabla \mathbf{u}) = \frac{\nabla \rho \times \nabla p}{\rho^3}. \tag{34}$$

We contemplate isolines of pressure and density.

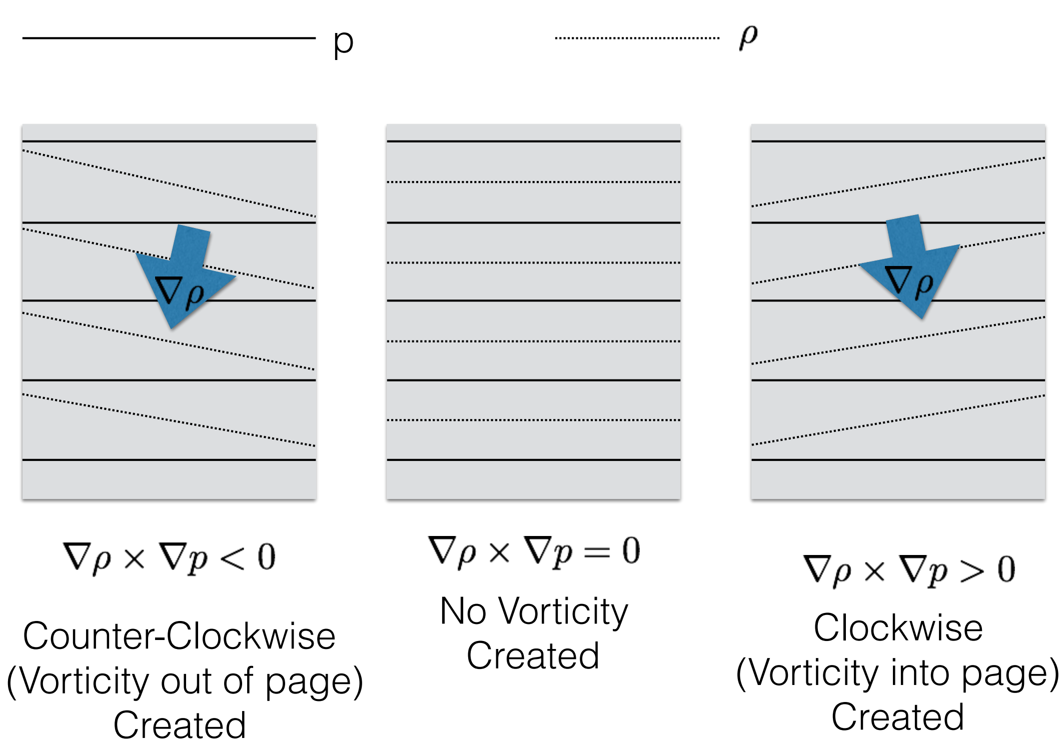

Figure 1: **Isolines of density and pressure reveal the direction of baroclinic vorticity generation.**

5 Problem 5: Polar PV

Consider (1.51) of section 1.5 of (Bühler, 2014). Suppose a fluid parcel is rotating together with the earth at the North Pole (f=$2 \cdot 2\pi/24$ hours, $\mathbf{u} = 0$), and that the local entropy, s, increases upward. Relocate this parcel, conserving its entropy, density, and potential vorticity to the South Pole (f=$-2 \cdot 2\pi/24$ hours), where again we assume that entropy increases upward (based on the local vertical), with the same gradient magnitude as at the North Pole. Describe the parcel's rotation relative to the earth.

We note that the vector $\mathbf{f}$ is always pointed in the same direction as the rotation axis of the earth, thus at the North Pole it is pointed vertically in the same direction as the entropy gradient, so $q = |f||\nabla s|$. At the South Pole $\mathbf{f}$ is pointed into the ground (locally downward in the vertical), and thus in the opposite direction of the locally upward entropy gradient. But, if PV is conserved, then q must be the same as before at the North Pole. The only way this occurs is if $|f||\nabla s| = (\nabla \times \mathbf{u} + \mathbf{f}) \cdot \nabla s = |2\mathbf{f} - \mathbf{f}||\nabla s|$, thus $\nabla \times \mathbf{u} = -2\mathbf{f}$, i.e., the parcel must be rotating relative to the earth anti-cyclonically (against the sense of rotation of the South Pole) with angular frequency $-2|f|$, or $\nabla \times \mathbf{u} = -2\mathbf{f}$.

6 Not required

Problem 2 of section 2.3 is interesting, if you are familiar with the calculus of variations and Hamiltonian methods.

References

Bühler, Oliver 2014 *Waves and mean flows*, 2nd edn. Cambridge, United Kingdom: Cambridge University Press.