1. Solution to ODEs

1.1 The phosphate cycle

Consider the box model of the phosphate cycle. The masses of the reservoirs are listed below:

1. Sediments = 4×10^9
2. Land = 2×10^5
3. Terrestrial biota = 3000
4. Oceanic biota = 138
5. Surface ocean = 2710
6. Deep ocean = 8.71×10^4

Regarding the given fluxes below, the system is at its steady state, i.e. at this point, masses of the reservoirs will stay constant and won’t change with time. (Fluxes have units of mass/year)

\[
\begin{align*}
F_{1-2} &= 20 \\
F_{2-1} &= 18.3 \\
F_{2-3} &= 63.5 \\
F_{2-4} &= 1.7 \\
F_{3-2} &= 63.5 \\
F_{4-3} &= 558 \\
F_{4-5} &= 42 \\
F_{5-4} &= 1040 \\
F_{5-6} &= 17.7 \\
F_{6-4} &= 1.7 \\
F_{6-5} &= 58
\end{align*}
\]

Use a linear approximation for the fluxes such that \( F_{A:B} = k_{A:B} A(t) \), where \( k \)'s are rate constants (1/time) and \( A(t) \) is the mass of phosphate in the reservoir \( A \) at time \( t \). Write the evolution of the mass of each reservoir over time as a set of linear ODEs, where \( \frac{dM}{dt} = KM \), where \( M \) is an array containing the mass of each reservoir at time \( t \) (dimensions 6x1) and \( K \) is a 6x6 matrix. Write down the matrix \( K \) in terms of rate constants \( k \)'s.

Decompose \( M \) into a basis generated from the eigenvector \( \nu_i \) of \( K \) and plug back into the system of differential equations to get an analytical solution for this box model.

Compare your analytical solution (plot it against ...) to the numerical solution from the code that we sent you. In that code, the Land reservoir is initially perturbed to 110% of its steady state mass. Try different initial perturbations (magnitude and reservoir) to check your analytical solution.

What do the eigenvalues of \( K \) represent? Discuss how you interpret them.
2.1 Forced harmonic oscillator

We will consider a damped harmonic oscillator with forcing

\[ m\ddot{x} + \mu \dot{x} + kx = F \cos wt, \]

where \( m \) is the mass of the oscillator, \( \mu \) a damping coefficient \( \mu \geq 0 \) and \( k \) the spring constant.

- What order is this ODE? Is it homogenous?
- Find the analytical solution to the non-damped case \( (\mu=0) \) for the case \( \frac{k}{m} \neq w^2 \).
- How many characteristic timescales do you have in this non-damped scenario? What are they?
- Consider now the damped case \( (\mu>0) \). Can you show that the generic function \( x(t) = A \cos w_0 t + B \sin w_0 t \), with \( w_0 = \sqrt{\frac{k}{m}} \), is no longer solution to the homogeneous damped oscillator equation.
- A choice for the homogenous solution to the damped oscillator equation is \( x(t) = Ae^{-Bt} \cos w_0 t \). Explain the logic behind the exponential prefactor, why does it make sense? This homogenous solution introduces a new timescale in the solution \( 1/B \), you can either derive mathematically the expression for \( B \) or BETTER use logic and your physics intuition to get it. How many characteristic timescales do we have in the problem?
- **Bonus**: Show that \( x(t) = G \cos wt + H \sin wt \) is a particular solution to the forced damped oscillator, what expression do you find for \( G \) and \( H \)? How does the damping term affect resonance?