Introduction

In this class, we learn how to apply proficiently concepts of maths & physics to your research.

Some of you are directly interested to study & solve equations such

$$\frac{\partial p}{\partial t} = \nabla \cdot \left( \frac{1}{\mu} \left( \frac{\partial \mathbf{u}}{\partial x} \right) \right)$$

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = \frac{1}{\partial x_j} \left[ \frac{1}{2} \epsilon_{ijk} \left( \frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k} \right) \right]$$

$$\frac{\partial C}{\partial t} = \nabla \cdot \left( D \nabla C \right) - \mathbf{k} \left( C - C_0 \right)$$

$$\left\{ \begin{align*}
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} & = - \nabla p + \rho \mathbf{g} + \nabla^2 \mathbf{u} \\
\nabla \cdot \mathbf{u} & = 0
\end{align*} \right\}$$

and many more...

- What do they have in common?

   - differential equations.

Physics is about predicting/extrapolating the state of a system and especially its variation with space & time!

Variations → introduce differential operators.
The first two modules of this class

1. Tensor calculus
2. Continuum mechanics

are designed to provide the tools to derive, use and solve differential equations that we commonly face in the Earth, Environmental & Planetary Science.

The language of physics is calculus....

Feynman asked once a journalist doing research on the Manhattan Project whether he knew calculus....

Feynman said "It's the language God talks"...

Calculus provides a language where differential equations are sentences to study & understand the world around us.

Einstein: "The eternal mystery of the world is its comprehensibility."

Wigner: "The miracle of the appropriateness of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve."

(see more in "Infinite Power" S. Strang)
Our roadmap:

(1) Fundamental theorem of calculus
   - What is it?
   - What does it mean?
   - Why should I care?

=> answer: in a nutshell it formalizes how one approaches solving physics problems:
   \( \Rightarrow \) derivation of differential equation (ex. statement of conservation law)
   \( \Rightarrow \) solution by integration.

(2) Continuous functions are vectors! Using some simple linear algebra to solve differential equations
   - What is a vector? A vectorial space?
   - How do you choose a vectorial basis for continuous functions?
     *(c)* Taylor series
     *(c)* Fourier series
     *(c)* Trigonometric series
     *(c)* Legendre polynomial / spherical harmonics
     *(c)* Bessel functions
(3) Differential operators & Integrable one linear operators

(4) From 1 to N Dimensions - Multivariate Calculus
   (a) Partial derivatives
   (b) Vector calculus, div, grad, curl
   (c) Tensors and some aspect of symmetry

(5) Continuum mechanics
   (a) Eulerian vs. Lagrangian reference frame
   (b) Gauss theorem, Reynolds transport theorem
   (c) Generic conservation laws (continuity, Cauchy...)

   (d) Closure equations
      (i) Linear transport theory
      (ii) intuition & symmetry

   (e) Buckingham & dimensional analysis
The Fundamental Theorem of Calculus

Idea: Integrals and derivatives are related to each other; they are opposites.

why do we care?

(a) derivatives are easy to compute, not the case for integrals → can use inference with guesses from our understanding of derivatives to solve integrals analytically.

\[ \int [3x^2 + 2x + 5] \, dx = ? \]

\[ \int 3x^2 \, dx + \int 2x \, dx + \int 5 \, dx \]

\[ \frac{d}{dx}(x^3) = 3x^2 \quad \frac{d}{dx}(x^2) = 2x \quad \frac{d}{dx}(5x) = 5 \]

\[ \int \frac{d}{dx}(x^3) \, dx + \int \frac{d}{dx}(x^2) \, dx + \int \frac{d}{dx}(5x) \, dx \]

\[ = x^3 + x^2 + 5x \quad \text{using fundamental th. of calculus}. \]

(b) \( \frac{M}{dx^2} \frac{dv}{dx} = \frac{dp}{dx} \quad \text{Stokes flow (unidirectional)} \)

\( \frac{dv}{dy} = 0 \quad \text{with incompressible fluid} \)

\( \Rightarrow \text{Differential Equations solving } p(x) \& V(x) \)

requires integration.
more generally, physical laws are expressed generally as differential equations, solution(s) require integration.

Functions of a single variable

\[ F(x) \] is an anti-derivative of \[ f(x) \] if

\[
\frac{d}{dx} F(x) = f(x)
\]

(there is an infinite number of them for any well-behaved \( f(x) \))

One statement of the Fundamental Theorem of Calculus reads

\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a)
\]

Let's build some intuition into it, assume you're running around on campus from position \( x = a \) to \( x = b \)

(we can assume a pseudo 1D trajectory for this idealized case)

There are hills & flat portions such that your instantaneous velocity \( v(x) \) can vary a lot. You want to compute your average velocity \( \langle v \rangle \)

We can use 2 approaches/points of view
how fast did I go? Well, I can use a GPS-running watch to measure my "instantaneous" velocity \( V(t) \) at different times.

\[
\langle V \rangle = \frac{1}{b-a} \int_{a}^{b} V(t) \, dt \quad (\ast)
\]

Alternatively, if I know the distance I ran \( \delta \) and the time it took \( (b-a) \) then

\[
\langle V \rangle = \frac{X(t=b) - X(t=a)}{b-a} \quad (\ast\ast)
\]

Equating \( (\ast) \) & \( (\ast\ast) \)

\[
\frac{1}{b-a} \int_{a}^{b} V(t) \, dt = \frac{X(t=b) - X(t=a)}{b-a}
\]

\[
= \int_{a}^{b} V(t) \, dt = X(t=b) - X(t=a)
\]

with \( X(t) \Rightarrow \text{related to } V(t) \text{ by} \)

\[
\frac{dx}{dt} = V(t) \quad (X \text{ is anti-derivative of } V)
\]
In a nutshell, here is the fundamental theorem of calculus:

\[ F(b) - F(a) = \int_{a}^{b} f(x) \, dx \quad \text{when} \quad \frac{dF}{dx} = f(x) \]

Let's go further:

\[ F(x) - F(a) = \int_{a}^{x} f(x') \, dx' \]

\[
\frac{d}{dx} \left( \int_{a}^{x} f(x') \, dx' \right) = \frac{dF(x)}{dx} = f(x) \quad \text{as we know} \quad \frac{d}{dx} \left[ \int_{a}^{x} f(x') \, dx' \right] = f(x), \quad \text{which is another way to frame the theorem.}
\]

What does it mean?

1. Derivatives & integrals are opposites.
2. Because they are opposite & derivatives are much easier to compute, we can compute integrals by guessing the correct anti-derivative...
(3) Every function $f(x)$ that can be written in terms of a primitive function $F(x)$ such that $\frac{dF}{dx} = f(x)$ is such that its integral (definite) is equal to the value of the difference of $F$ at the 2 bounds. (b & a).

In more than 1-D if $f(x)$ is a function (here a scalar function for the sake of simplicity) of multiple variables

\[ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ position} \; ; \; \bar{x} = \begin{pmatrix} p \\ T \end{pmatrix} \text{ thermodynamic} \; \begin{pmatrix} \text{pressure} \\ \text{temperature} \end{pmatrix} \]

Intuitively, the fundamental theorem would suggest that

\[ F(\bar{x}_b) - F(\bar{x}_a) = \int_{\bar{x}_a}^{\bar{x}_b} f(\bar{x}) \, d\bar{x} \]

Well that would be great because it would mean

(a) that the integral is independent of the path $\gamma$ as long as it starts from $\bar{x}_a$ and ends at $\bar{x}_b$

(b) if $\bar{x}_b = \bar{x}_a \Rightarrow (1)$ implies that for a closed path

\[ \oint f(\bar{x}) \, d\bar{x} = 0 \]
For a 1-D function $f(x)$, we saw that the condition is that

$$\frac{dF}{dx} = f(x) \quad F = \text{anti-derivative of } f$$

What condition(s) need(s) to be fulfilled in more dimensions with $f(\vec{x})$ & $F(\vec{x})$?

We will use 2 examples drawn from physics:

(A) thermodynamics - Gibbs free energy

(B) Gravity versus friction
   (conservative)    (non-conservative)

(A) Path invariance in thermodynamics

Assume a closed system is initially in state $A$ given by $(T_A, p_A)$ (temperature/pressure) and is changing to a state $B$ defined by $(T_B, p_B)$, then the change in Gibbs free energy experienced by the system is independent of the $(T, p)$ path followed to go from $A \rightarrow B$. 

\[ C \quad \bullet \quad B \]

\[ T \]

\[ A \quad \bullet \quad p \]
Let's define a change in Gibbs free energy

$$dG = \alpha(T, \rho) \, dT + \beta(T, \rho) \, d\rho$$

where \( \alpha, \beta \) are functions that will assume "special" properties to ensure path independence.

Let's define a transition state \( C \rightarrow (T_c, \rho_c = \rho_A) \) along the path from \( A \rightarrow C \)

$$dG = \alpha(T, \rho) \, dT$$, because \( d\rho = 0 \)

$$\Delta_{A \rightarrow C} G = \int_{T_A}^{T_C} \alpha(T, \rho) \, dT$$

\( \equiv dG \) at fixed pressure

Similarly, \( \Delta_{C \rightarrow B} G = \int_{p_c}^{p_B} (dG)_{\text{fixed } T} = \int_{p_c}^{p_B} \beta(T, \rho) \, d\rho \)

Path independence

$$\Delta_{A \rightarrow B} G = \int_{A}^{T_C} \int_{T_A}^{p_B} \alpha(T, \rho) \, dT \, d\rho$$

It works only if \( \alpha(T, \rho) = \left( \frac{\partial G}{\partial T} \right)_{\text{fixed } \rho} \) and \( \beta = \left( \frac{\partial G}{\partial \rho} \right)_{\text{fixed } T} \)

\( \Rightarrow \) path independence when integrating a multivariate
function like Gibbs free energy changes $dG$ is satisfied if and only if

$$dG(x, y, z) = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial y} dy + \frac{\partial G}{\partial z} dz = \nabla G \cdot dx$$

and

$$\nabla G = \left( \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}, \frac{\partial G}{\partial z} \right)$$

in that case $dG$ is said to be a total derivative.

And

$$G(\mathbf{x}_B) - G(\mathbf{x}_A) = \int_{\mathbf{A}}^{\mathbf{B}} \nabla G \cdot dx$$

in that case $dG$ is said to be a total derivative and

for functions such as $dG$ (total derivative)

$$\int_{\mathbf{A}}^{\mathbf{B}} \nabla G \cdot dx = 0$$

over a closed path $\mathbf{A} \rightarrow \mathbf{B} \rightarrow \mathbf{A}$

so

$$G(\mathbf{x}_B) - G(\mathbf{x}_A) = \int_{\mathbf{A}}^{\mathbf{B}} \mathbf{f} \cdot dx$$

works if $\mathbf{f}$ is the gradient of $G$

$$\mathbf{f} = \left( \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}, \frac{\partial G}{\partial z} \right)$$

Interesting properties of total derivatives:
\[
dG = \frac{\partial G}{\partial T} dT + \frac{\partial G}{\partial \rho} d\rho \quad \text{for example}
\]

\[
\frac{\partial \alpha}{\partial \rho} = \frac{\partial \beta}{\partial T} \quad \text{(simple check)}
\]

Because
\[
\frac{\partial \alpha}{\partial \rho} = \frac{\partial}{\partial \rho} \left( \frac{2G}{\partial T} \right) = \frac{2}{\partial T} \left( \frac{2G}{\partial \rho} \right) \quad \text{(exchange of limits)}
\]
\[
\frac{\partial \beta}{\partial T} = \frac{2}{\partial T} \left( \frac{2G}{\partial \rho} \right)
\]

\[\Rightarrow \text{Maxwell's relations in thermodynamics}\]
\[
dG = - S dT + V d\rho \quad \text{(fixed mass)}
\]

\[
(\alpha = -S; \quad \beta = V) \quad \text{thermal expansion coefficient}
\]
\[
\Rightarrow \quad - \left( \frac{\partial S}{\partial \rho} \right)_T = \left( \frac{\partial V}{\partial T} \right)_\rho = + \gamma_T V
\]

Little game: What is the temperature profile \( T \) in a connecting fluid where adiabatic fluid ascent is assumed?

adiabatic means no heat exchange with surrounding

\[
\delta Q = T dS = c_p dT = 0 \quad \text{we wait some}
\]

\[
\left( \frac{\partial T}{\partial S} \right)_\rho = \frac{1}{c_p}
\]
\[
\left( \frac{\partial T}{\partial p} \right)_s = - \left( \frac{\partial T}{\partial s} \right)_p \left( \frac{\partial s}{\partial p} \right)_T = \frac{\Delta T}{C_p} V
\]

\(dG\) is expressed as an energy/unit mass

So \(V\) is a volume/unit mass

\[V = \frac{1}{\rho}\]

\[
\left( \frac{\partial T}{\partial p} \right)_s = \frac{\Delta T}{\rho C_p}
\]

is the adiabatic \(T\) profile in a fluid

\[
\left( \frac{\partial T}{\partial z} \right)_s = \left( \frac{\partial T}{\partial p} \right)_s \left( \frac{\partial p}{\partial z} \right)_s
\]

if pressure profile is near hydrostatic \(\ldots\)

(can it be hydrostatic?)

\[
\Rightarrow \frac{\partial p}{\partial z} \sim \rho \phi
\]

\[
\Rightarrow \left( \frac{\partial T}{\partial z} \right)_s \sim \frac{\Delta T}{C_p} \rho \phi
\]
B  Gravity a conservative force

Conservative forces are often said to "derive" from a potential \( V(\mathbf{x}) \) via gradient

so if \( \mathbf{f} = -\nabla V \) is a force associated with potential \( V \)

\[
\nabla V = \begin{pmatrix}
\frac{\partial V}{\partial x} \\
\frac{\partial V}{\partial y} \\
\frac{\partial V}{\partial z}
\end{pmatrix}
\]

such a force is said to be conservative if work around closed path is zero

\[
\int_C \mathbf{f} \cdot d\mathbf{r} = 0
\]

check \( -\int \nabla V \cdot d\mathbf{r} = V(x_0) - V(x_0) = 0 \)

Fundamental th. of calculus.

e.g. Gravity force \( \mathbf{F}_g = -m \nabla V_g \)

where \( V_g(\mathbf{x}) \) is the gravitational potential

\( V_g(\mathbf{x}) \) is or using radial (spherical) coordinates centered on the object "creating" the potential

\( V_g(r) = -\frac{GM}{r} \) \( \rightarrow \mathbf{F}_g = -\frac{GMm}{r^2} \)

\[
\int \mathbf{F}_g \cdot d\mathbf{r} = -m \int \nabla V_g \cdot d\mathbf{r} = -m \left[ V_g(x_{\text{end}}) - V_g(x_0) \right]
\]

= 0 on closed loop
Now, let's consider friction (linear)

\[ \vec{f} = -\mu \frac{d\vec{x}}{dt} \quad (\text{-}\mu \vec{v}) \]

Let's assume that we follow a circular trajectory and compute the net work performed by this force.

\[ W = \int \vec{f} \cdot d\vec{l} \quad , \quad d\vec{l} = \frac{d\vec{x}}{dt} \, dt \]

\[ x = \cos (2\pi t) \quad \Rightarrow \quad \frac{d\vec{x}}{dt} = \left( -2\pi \sin (2\pi t), \quad 2\pi \cos (2\pi t) \right) \]

\[ \int_0^1 d\vec{l} \rightarrow \int_0^1 \frac{d\vec{x}}{dt} \, dt \]

\[ W = -\int_0^1 \mu \frac{d\vec{x}}{dt} \cdot \frac{d\vec{x}}{dt} \, dt = -\mu 4\pi^2 \int_0^1 \left[ \sin^2 (2\pi t) + \cos^2 (2\pi t) \right] dt \]

\[ = -4\pi^2 \mu \neq 0 \quad \Rightarrow \text{not a conservative force} \]

\[ \vec{f} \text{ cannot be written as the gradient of a potential} \]

\[ \vec{f} \neq \nabla \Psi \Rightarrow \text{so} \]

Fundamental theorem of calculus doesn't apply.