Back to Christoffel Symbols

\[ \Gamma^e_{ij} = \frac{\partial \hat{e}^e}{\partial x^j} \]

\[ \Gamma^e_{ijk} = g^{ek} \Gamma^e_{ij} \quad \text{&} \quad \Gamma^e_{ij} = g^{ek} \Gamma^e_{ij} \]

Because we have

\[ g_{ij} = \hat{e}^i \cdot \hat{e}^j \]

then

\[ \frac{\partial g_{ij}}{\partial x^k} = \frac{\partial (\hat{e}^i \cdot \hat{e}^j)}{\partial x^k} = \frac{\partial \hat{e}^i}{\partial x^k} \hat{e}^j + \frac{\partial \hat{e}^j}{\partial x^k} \hat{e}^i \]

Remembering that \( \Gamma^e_{ijk} = g^{ek} \frac{\partial \hat{e}^i}{\partial x^j} = \frac{\partial \hat{e}^i}{\partial x^j} \hat{e}^e \)

\[ \frac{\partial g_{ij}}{\partial x^k} = \Gamma^e_{ijk} + \Gamma^e_{jki} \]

& playing with indices leads similarly to:

\[ \frac{\partial g_{ki}}{\partial x^j} = \Gamma^e_{kji} + \Gamma^e_{ijk} \]

& \[ \frac{\partial g_{ik}}{\partial x^j} = \Gamma^e_{jik} + \Gamma^e_{kij} \]

\[ \Rightarrow \Gamma^e_{ijk} = \left[ \frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right] / 2 \]
and therefore
\[ \Gamma^e_{ij} = g^{ek} \Gamma_{ijk} = \frac{g^{ek}}{2} \left[ \frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right] \]

So we have seen the covariant derivative of a contravariant vector
\[ \vec{A} = A^i \hat{e}_i \]

\[ \frac{\partial \vec{A}}{\partial x^i} = \left( \frac{\partial A^i}{\partial x^i} + \Gamma^i_{jk} A^k \right) \hat{e}_i = \nabla_j A^i \hat{e}_i \]

\[ = \nabla_j A^i \Rightarrow \text{this transforms as a mixed tensor.} \]

The covariant derivative of a covariant vector:
\[ \vec{A} = A_i \hat{e}_i \]

\[ \frac{\partial A_i \hat{e}_i}{\partial x^j} = \frac{\partial A_i}{\partial x^j} \hat{e}_i + \frac{\partial \hat{e}_i}{\partial x^j} A_i \]

\[ = \Gamma^k_{ij} \hat{e}_k \]

\[ \Rightarrow \nabla_j A_i = \frac{\partial A_i}{\partial x^j} - \Gamma^k_{ij} A_k \]
Oh, let's pause an instant & summarize.

In vector calculus, one can compute the gradient of a vector and obtain "an object" of rank 2 (a matrix)
\[ \nabla \vec{A} = \nabla \vec{M} \quad \text{if} \quad \vec{A} \in \mathbb{R}^n, \text{ then } \vec{M} \in \mathbb{R}^{n \times n} \]

In tensor calculus, the invariance of physical laws to coordinate changes influences how one thinks about differentiation (grad, div...).

\[ \text{ex.} \quad \nabla \cdot \vec{A}^i = \frac{\partial A^i}{\partial x^j} + \Gamma^i_{jk} A^k \]

\[ \text{with } \quad \Gamma^i_{jk} = \frac{1}{2} g^{im} \left[ \frac{\partial g_{km}}{\partial x^j} + \frac{\partial g_{mj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^m} \right] \]

\[ \Rightarrow \text{ the new term } \neq 0 \text{ only if metric is not homogeneous! (Deformed space!)} \]

\[ \Rightarrow \text{ live in Cartesian or Std spherical coordinates.} \]

Let's go back to our linear algebra description where we learned about dual space \( E^* \), with \( \vec{W} \in E^* \)

\[ \vec{W} = W^\mu \hat{x}^\mu \quad \text{coordinates } W^\mu \text{ are covariant} \]

\[ \downharpoonleft x \in E \quad W^\mu = W^\mu (x) \text{ is a 1-form (linear)} \]

\[ \hat{x}^\mu \text{ is a contravariant basis of } E^* \]
There is plenty of choice for the basis $\mathcal{X}^m$:

$$W_\mu (x^\nu) = dx^\mu$$

the differential form (coordinate) is a good choice for a basis if $\hat{e}_\mu$ are linearly independent $\Rightarrow dx^\mu \mid_{\mu=1,...,N}$ are also linearly independent.

$$\hat{x}^m \rightarrow dx^m$$ will be our choice of basis for $E^*$.

So any vector $\vec{w} \in E^*$ can be written (unique) as:

\[
\vec{w} = \sum_{\mu} f_\mu \; dx^\mu \quad \text{with} \quad f_\mu \quad (W_\mu \rightarrow f_\mu)
\]

In Cartesian coordinates a 1-form can be written

$$w = f_x \; dx + f_y \; dy + f_z \; dz$$
A staple of vectorial/homological calculus is Stokes' theorem: if the outer derivative of \( w \rightarrow dw \) & \( Y \) is the geometric border of an object \( Y \) \( (Y = \partial Y) \)

Stokes th: \( \int_Y w = \int_{\partial Y} dw \)

For 0-form \( w = f(\mathbf{x}) \) scalar field in \( E^n \)

Stokes theorem \( \rightarrow \int_Y \nabla f \cdot d\mathbf{e} = f(\mathbf{x}(1)) - f(\mathbf{x}(0)) \)

Gradient theorem.

The outer derivative \( dw \) of \( w \) is an interesting object.

If \( w \) is a k-form \( (\text{e.g. } E^n = \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n) \)

ex. 2-form will be an operator with 2 indices (hydraulic permeability has 2 indices)

then \( dw \) is a \((k+1)\)-form

Here, we will be interested by 2-forms and will write a 2-form

\[ w_2 = W_{i,j}(\mathbf{x}) \, dx^i \wedge dx^j \]

think of a 2nd rank basis vector

but akin to cross-product \( \wedge \) anti-symmetric
\[ dx^i \wedge dx^i = - dx^j \wedge dx^i \]

for 3-form

\[ w_3 = w_{ijk} \hat{x}^i dx^j \wedge dx^k \quad (\text{still anti-symmetric}) \]

Ok so 1-form \( w = f_x dx + f_y dy + f_z dz \)

\[ dw = df_x \wedge dx + df_y \wedge dy + df_z \wedge dz \]

with \( df_x = \frac{\partial f_x}{\partial x} dx + \frac{\partial f_x}{\partial y} dy + \frac{\partial f_x}{\partial z} dz \)

\[ \Rightarrow dw = \left( \frac{\partial f_x}{\partial y} \right) dx \wedge dy + \left( \frac{\partial f_x}{\partial z} \right) dx \wedge dz + \left( \frac{\partial f_y}{\partial x} \right) dy \wedge dx + \left( \frac{\partial f_y}{\partial y} \right) dy \wedge dy + \left( \frac{\partial f_y}{\partial z} \right) dy \wedge dz + \left( \frac{\partial f_z}{\partial x} \right) dz \wedge dx + \left( \frac{\partial f_z}{\partial y} \right) dz \wedge dy + \left( \frac{\partial f_z}{\partial z} \right) dz \wedge dz \]

Regrouping & using \( dx \wedge dy = - dy \wedge dx \)

\[ dw = \left( \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right) dx \wedge dy + \left( \frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z} \right) dy \wedge dz + \left( \frac{\partial f_z}{\partial x} - \frac{\partial f_x}{\partial z} \right) dz \wedge dx \]

\[ = \nabla \times F \cdot d^2 x \]
Back into Stokes

\[ \int \int dw = \int f \, \, d^2x \]

\[ \int \nabla \times \mathbf{f} \cdot d^2x = \oint \mathbf{f} \cdot d\mathbf{l} \quad \text{Green's th.} \]

\[ \text{if} \quad w \quad \text{is a 2-form} \]

\[ \mathbf{W} = f_x \, dy \wedge dz + f_y \, dz \wedge dx + f_z \, dx \wedge dy \]

\[ dw = df_x \wedge dy \wedge dz \quad \text{with} \quad df_x = \frac{\partial f_x}{\partial x} \, dx + \frac{\partial f_x}{\partial y} \, dy + \frac{\partial f_x}{\partial z} \, dz \]

\[ dw = \left( \frac{\partial f_x}{\partial x} \, dx + \frac{\partial f_x}{\partial y} \, dy + \frac{\partial f_x}{\partial z} \, dz \right) \wedge dy \wedge dz + \ldots \]

Wait a 2nd, \( \alpha \, dy \wedge dy \wedge dz = \beta \, dz \wedge dy \wedge dz \) because of anti-symmetric nature of 2-forms.

3-forms

\[ \Rightarrow \quad dw = \left( \frac{\partial f_x}{\partial x} \right) \, dx \wedge dy \wedge dz + \left( \frac{\partial f_y}{\partial y} \right) \, dx \wedge dy \wedge dz + \left( \frac{\partial f_z}{\partial z} \right) \, dx \wedge dy \wedge dz = \nabla \cdot \mathbf{f} \, dx \wedge dy \wedge dz \]
So, Stokes' Theorem

\[ \int_{\mathbf{V}} \nabla \cdot \mathbf{F} \, d\mathbf{V} = \oint_{\mathbf{S}} \mathbf{F} \cdot d\mathbf{S} \quad \text{Gauss' Theorem} \]

So, the tools of vectorial calculus work with tensors, the definition of grad, div, curl is a bit different if

\[ \frac{\partial g_{ij}}{\partial x_k} \neq 0 \quad \text{though.} \]
First and eulerian approach to conservation law

Let $f_u(x)$ be a scalar quantity that is conserved (mass density, energy density) associated with a flux $\overrightarrow{q}_u(x)$.

The total amount of "u" in volume $V$ is at time $t$:

$$\int f_u(x,t) \, dV$$

at a late time $t + \Delta t$, it is:

$$\int f_u(x,t+\Delta t) \, dV$$

for $V(t+\Delta t)$.

Now, we will start with a Lagrangian eulerian proof where $V(t+\Delta t) = V(t)$, fixed reference volume.
The variation of mass/energy (rate of change) in $V$

$$\frac{d}{dt} \int p_u(x, t) \, dV = \int \frac{\partial p_u}{\partial t} \, dV$$

If we now consider flux $\vec{q}_u$ the net change in mass/energy in $V$ can only be caused by unbalanced fluxes $\vec{q}_u \Rightarrow \oint \vec{q}_u \cdot \hat{n} \, ds$

or Source/Sink terms (here per unit volume)

$$\oint \vec{S} \, dV$$

$$\frac{d}{dt} \int p_u(x, t) \, dV = \int \frac{\partial p_u}{\partial t} \, dV = \oint \vec{q}_u \cdot \hat{n} \, ds + \oint \vec{S} \, dV$$

Gauss th.: $\oint \vec{q}_u \cdot \hat{n} \, ds = \int \nabla \cdot \vec{q}_u \, dV$
\[\Rightarrow \int \left[ \frac{\partial q_u}{\partial t} + \nabla \cdot q_u - q \right] dV = 0\]

\[\Rightarrow \left[ - - - \right] = 0\]

\[\Rightarrow \frac{\partial q_u}{\partial t} + \nabla \cdot q_u = q\]

Assume \(q = 0\) \& a velocity field \(\vec{V}\) carries \(q_u\) at flux \(\vec{q}_u = q_u \vec{V}\)

\[\Rightarrow \frac{\partial q_u}{\partial t} + \nabla \cdot \left( q_u \vec{V} \right) = 0 \Rightarrow \text{continuity equation}\]