Scale-aware subgrid closures for models that partly resolve the mesoscale and submesoscale
The Earth’s Climate System is driven by the Sun’s light (minus outgoing infrared) on a global scale.

Dissipation concludes turbulence cascades to scales about a billion times smaller.
Air-Sea Flux Errors vs. Data

Heat capacity & mode of transport is different in A vs. O

Resolution will be an issue for centuries to come!

If we can’t resolve a process, we need to develop a parameterization or subgrid model of its effect.
Big, Deep (mesoscale) interact with Little, Shallow (submeso)

The Character of the Submesoscale

(Capet et al., 2008)

- Fronts
- Eddies
- Ro=O(1)
- Ri=O(1)
- near-surface
- 1-10km, days

Eddy processes often baroclinic instability

Parameterizations of submesoscale baroclinic instability?


Physical Sensitivity of Ocean Climate to MLE: (submeso) Mixed Layer Eddy Restratification Improves CFCs (water masses)

A consistently restratifying, and horizontally downgradient flux.

\[
\bar{u}'_H \bar{b}' \propto -\frac{H^2}{|f|} \frac{\partial \bar{b}}{\partial z} \nabla_H \bar{b}
\]

and horizontally downgradient flux.

\[
\bar{w}' \bar{b}' \propto \frac{H^2}{|f|} \left| \nabla_H \bar{b} \right|^2
\]
Mixed Layer Problem--Southern Ocean too shallow!

What's missing?


Sallee et al. (2013) have shown that a too shallow S. Ocean MLD is true of most* present climate models salinity forcing or ocean physics?

*true for CMIP5 multi-model ensemble

Shallow ML Bias worse
Lesson: We can study a small-scale system (1-10km submeso mixed layer eddies), derive parameterizations, and then use them to improve climate models & assess impact globally.

This particular process relied heavily on thermal wind (geostrophic & hydrostatic) scaling relationships.

Corollary: But, what about things we haven't thought of yet? e.g., things that aren't geostrophic & hydrostatic?

For example, waves and near-surface 3d turbulence.
Waves, waves, waves

I will discuss surface wave effects on upper ocean physics on larger & slower scales.

- **On Langmuir Turbulence Scales**
  - (10–100m, 10–100min)

- **Submesoscales**
  - (1–10km, 0.1 to 10 days)

- One test involving Langmuir–Submesoscale coupling (10m–10km, 30 days)
Surface Wave Primer

Look for fast, small solutions of the Boussinesq Equations:

The irrotational, incompressible flow obeys

$$
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0
$$

The boundary conditions are:

- **Solid Bottom**
  \[ w = \frac{\partial \phi}{\partial z} = 0 \] at \( z = -H \)

- **Pressure Matching (dynamic)**
  \[ p = 0 \] at \( z = \eta \)

- **Velocity Matching (kinematic)**
  \[ \frac{D \eta}{Dt} = w_\eta \] at \( z = \eta \)

Illustration of wave spectra from different types of ocean surface waves (Holthuijsen, 2007)
Surface Wave Primer

Look for fast, small solutions of the Boussinesq Equations:

Linearized for not steep waves

The irrotational, incompressible flow obeys

\[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \]

The boundary conditions are (small steepness):

Solid Bottom

\[ w = \frac{\partial \phi}{\partial z} = 0 \quad \text{at} \quad z = -H \]

Pressure Matching (dynamic)

\[ \frac{\partial \phi}{\partial t} = -g \eta \quad \text{at} \quad z = 0 \]

Velocity Matching (kinematic)

\[ \frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} \quad \text{at} \quad z = 0 \]
Particle motions

The $u$, $v$, decay exponentially toward the bottom with decay scale proportional to the wavelength.

Thus, $kH$ is a measure of depth

$ka$ is a measure of steepness

Deep water waves don't "feel" the bottom. Implies nonhydrostatic ($Ro \gg 1$) & fast timescale.
Formally a multiscale asymptotic equation set:
- 3 classes: Small, Fast; Large, Fast; Large, Slow
- Solve first 2 types of motion in the case of limited slope (ka), irrotational --> Deep Water Waves!
- Must also assume slowly-varying wave packets
- Average over deep water waves in space & time,
- Arrive at Large, Slow equation set:

$$\frac{\partial v}{\partial t} + [f + \nabla \times v] \times (v + v_s) = -\nabla \pi^\dagger + bk + \nu \nabla^2 v$$

$$\frac{\partial b}{\partial t} + (v + v_s) \cdot \nabla b = 0$$
$$\nabla \cdot v = 0$$

$v_s =$ Stokes Drift

Craik & Leibovich 1976; Gjaja & Holm 1996; McWilliams et al. 2004
What is Stokes Drift?

Take wave solns, compare the velocity of trajectories vs. Eulerian velocity, Taylor Expand, calculate:

\[ u^L(x_p(t_0), t) - u^E(x_p(t_0), t) \approx [x_p(t) - x_p(t_0)] \cdot \nabla u^E(x_p(t_0), t) \]
\[ \approx \left[ \int_{t_0}^{t} u^E(x_p(t_0), s')ds' \right] \cdot \nabla u^E(x_p(t_0), t). \]

Examples:

Monochromatic:

\[ u^S = \hat{e}^w \frac{8\pi^3 a^2 f^3 p}{g} e^{\frac{8\pi^2 f^2}{g^2 z}} = \hat{e}^w a^2 \sqrt{gk^3} e^{2kz}. \]

Spectrum:

\[ u^S = \frac{16\pi^3}{g} \int_{0}^{\infty} \int_{-\pi}^{\pi} (\cos \theta, \sin \theta, 0)f^3 S_{f\theta}(f, \theta)e^{\frac{8\pi^2 f^2}{g^2 z}}d\theta df. \]
How well do we know Stokes Drift? <50% discrepancy

RMS error in measures of surface Stokes drift, 2 wave models (left), model vs. altimeter (right)

Year 2000 data & models

The Character of the Langmuir Scale

- Near-surface
- Ro >> 1
- Aspect O(1): Nonhydro
- 1-10m
- 10s to mins
- w, u = O(10 cm/s)
- Stokes drift
- Eqtns: Craik-Leibovich
- Params: McWilliams & Sullivan, 2000, etc.

Figure 1 Sketch showing the pattern of mean flow in idealized Langmuir circulation. The windrows may be 2 m to 300 m apart, and the cell form is roughly square (as shown). In practice the flow is turbulent, especially near the water surface, and the windrows (Figure 2) amalgamate and meander in space and time. Bands of bubbles or buoyant algae may form within the downward-going (or downwelling) flow (see Figure 3).
What's plotted are surfaces of large vert. velocity

wind & wave direction

depth (m)

x (km)

y (km)
CLB as equations for Large Eddy Simulations: Tricky: Misaligned Wind & Waves

Tricky: Misaligned Wind & Waves

Tricky: Misaligned Wind & Waves

Tricky: Misaligned Wind & Waves

Why? Vortex Tilting Mechanism

In CLB: Tilting occurs in direction of $u_L = v + v_s$

\[
\frac{\partial \xi}{\partial t} + (u_L \cdot \nabla) \xi = (\omega \cdot \nabla)(u_L \cdot \hat{x}') + (\nabla b \times \hat{z}) \cdot \hat{x}' + SGS,
\]

**Figure 1** Sketch showing the pattern of mean flow in idealized Langmuir circulation. The windrows may be 2 m to 300 m apart, and the cell form is roughly square (as shown). In practice the flow is turbulent, especially near the water surface, and the windrows (Figure 2) amalgamate and meander in space and time. Bands of bubbles or buoyant algae may form within the downward-going (or downwelling) flow (see Figure 3).
Generalized Turbulent Langmuir No., Projection of $u^*$, $u_s$ into Langmuir Direction

A scaling for LC strength & direction!

Why? Vortex Tilting Mechanism

Misalignment enhances degree of wave-driven LT

In CLB: Tilting occurs in direction of $u_L = v + v_s$

Physical Model by N. Suzuki (Brown)

\[
\begin{align*}
\partial_t u + (\vec{u}^L \cdot \nabla) u &= -\partial_x \tilde{p} + f v^L \\
\partial_t v + (\vec{u}^L \cdot \nabla) v &= -\partial_y \tilde{p} - f u^L \\
\partial_t w + (\vec{u}^L \cdot \nabla) w &= -\partial_z \tilde{p} + \tilde{b} - (u', v') \cdot \partial_z (u^S, v^S)
\end{align*}
\]

Stokes-shear force
- solely responsible for the CL2 instability
- conditional: acts only on \((u', v')\)
- directional:

\[\langle (u' w', v' w') \rangle \cdot \partial_z (u^S, v^S)\]

\[\begin{array}{c}
0 \quad \text{pushed down} \\
\text{pushed up} \\
\text{weakly pushed down}
\end{array}\]
Direct influence on shear turbulence

Enhance the shear turbulence
Direct influence on shear turbulence

Kills the shear turbulence
But, does Langmuir Turbulence Matter?

Langmuir turbulence can only matter, in climate modeling practice, when winds and waves are not in equilibrium.

In this case, just knowing the winds is *insufficient* to predict the rate of Boundary Layer Mixing.

Thus, to do Langmuir mixing right, we need a wave model in addition to Atmosphere & Ocean.

But, in the meantime, we can use offline estimates using data...
Data + LES, Southern Ocean mixing energy: Langmuir (Stokes-drift-driven) and Convective

So, waves can drive mixing via Stokes drift (combines with cooling & winds)
Including Wave-driven Mixing (Harcourt 2013 parameterization) Deepens the Mixed Layer!

Conclusions on wave effects on Langmuir Scale

- Wave forced turbulence is an important contributor to boundary layer mixing.
- Wave effects are particularly needed in climate models to have scenarios where waves and winds are not in equilibrium, but this may require a prognostic wave model as a climate model component.
- Reducing the Southern Ocean mixed layer bias is a key deliverable of this effort.
The Character of the Submesoscale

(Capet et al., 2008)

- Fronts
- Eddies
- Ro=O(1)
- Ri=O(1)
- near-surface
- 1-10km, days

Eddy processes often baroclinic instability

Parameterizations of submesoscale baroclinic instability?


Geostrophy, Hydrostasy, & Thermal Wind

Traditional Mesoscale & Weak Submesoscale Oceanography inhabits a special distinguished limit:
Inviscid (Re>>1), rapidly rotating (Ro<1), and thin* (L>>H)

Full Momentum

\[
\frac{Dv}{Dt} + f \times v = -\nabla \phi + bk + \nu \nabla^2 v
\]

*\[
Re = \frac{UL}{\nu} \quad Ro = \frac{U}{fL} \quad Ri \equiv \frac{\partial b}{\partial z} \left( \frac{\partial u}{\partial z} \right)^2 \quad \alpha = H/L
\]

*closely related to strong stratification & ocean dimensions
Geostrophy, Hydrostasy, & Thermal Wind

Traditional Mesoscale & Weak Submesoscale Oceanography inhabits a special distinguished limit:
Inviscid (Re>>1), rapidly rotating (Ro<1), and thin* (L>>H)

(Horizontal) Geostrophic Balance
\[
\frac{Dv}{Dt} + \mathbf{f} \times \mathbf{v} = -\nabla \phi + bk + \nu \nabla^2 \mathbf{v}
\]

\[
Re = \frac{UL}{\nu} \quad Ro = \frac{U}{fL} \quad Ri \equiv \frac{\frac{\partial b}{\partial z}}{\left(\frac{\partial u}{\partial z}\right)^2} \quad \alpha = \frac{H}{L}
\]

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Geostrophy, Hydrostasy, & Thermal Wind

Traditional Mesoscale & Weak Submesoscale Oceanography
inhabits a special distinguished limit:
Inviscid (Re>>1), rapidly rotating (Ro<1), and thin* (L>>H)

(Vertical) Hydrostatic Balance

\[
\frac{D\mathbf{v}}{Dt} + \mathbf{f} \times \mathbf{v} = -\nabla \phi + b\mathbf{k} + \nu \nabla^2 \mathbf{v}
\]

\[
Re = \frac{UL}{\nu} \quad Ro = \frac{U}{fL} \quad Ri \equiv \frac{\partial b}{\partial z} \left( \frac{\partial u}{\partial z} \right)^2 \quad \alpha = \frac{H}{L}
\]

*closey related to strong statification & ocean dimensions
Geostrophy, Hydrostasy, & Thermal Wind

Traditional Mesoscale & Weak Submesoscale Oceanography inhabits a special distinguished limit:
Inviscid (Re >> 1), rapidly rotating (Ro < 1), and thin* (L >> H)

(Combined) Thermal Wind Balance

\[ f \times \frac{\partial v}{\partial z} = -\nabla b \]

Taken together with the forcing (air-sea) of buoyancy and the advection of buoyancy by this flow--you have the tools to study large-scale ocean physics!
Craik–Leibovich Boussinesq

Do waves affect the (sub)mesoscale? Yes!!


\[ \frac{\partial \mathbf{v}}{\partial t} + [\mathbf{f} + \nabla \times \mathbf{v}] \times (\mathbf{v} + \mathbf{v}_s) = -\nabla \pi^\dagger + b \mathbf{k} + \nu \nabla^2 \mathbf{v} \]

\[ \frac{\partial \mathbf{b}}{\partial t} + (\mathbf{v} + \mathbf{v}_s) \cdot \nabla \mathbf{b} = 0 \]

\[ \nabla \cdot \mathbf{v} = 0 \]

\[ \mathbf{v}_s = \text{Stokes Drift} \]

Craik & Leibovich 1976; Gjaja & Holm 1996; McWilliams et al. 2004
Now, Craik-Leibovich Boussinesq Equivalent:

(Combined) Lagrangian Thermal Wind Balance

\[ f \times \frac{\partial}{\partial z} (v + v_s) = f \times \frac{\partial v_L}{\partial z} = -\nabla b \]

Now the temperature gradients govern the Lagrangian flow, not the Eulerian!

Leading order consequence for small Rossby:

Anti-Stokes Effect:

Any Stokes drift that is unbalanced will provoke an Eulerian current to cancel it out!

So, can we just forget the whole thing and interpret large scales as Lagrangian velocities?

\[ [f + \nabla \times \mathbf{v}] \times \frac{\partial}{\partial z} (\mathbf{v} + \mathbf{v}_s) = -\nabla b \]

Not quite, because Ro>0 corrections are different!

The “Ro” for waves, is big *more often* than Ro is, especially for wide, shallow currents in a mixed layer.

Waves (Stokes Drift Vortex Force) \( \rightarrow \) Submeso, Meso: An example

Initial Submeso Front

Contours: 0.1

Perturbation on that scale due to waves

Contours: 0.014

Direct influence on the thermal wind


Stokes drift direction

Physical Model by N. Suzuki (Brown)
Direct influence on the thermal wind

Stokes drift direction

\[\times\]
Direct influence on the thermal wind
Direct influence on the thermal wind
Direct influence on the thermal wind

Stokes drift direction

[Diagram showing the direction of Stokes drift with blue and red regions and X marks indicating the drift direction.]
Conclusions on wave effects on (sub)mesoscale

- Wave forces significantly affect the dominant (sub)mesoscale balances in many places.
- The primary effect is Anti-Stokes Flow.
- The secondary effect is a Stokes vortex/Stokes shear force effect that disturbs hydrostatic & geostrophic balances.
What about Langmuir–Submeso Interactions?

Perform large eddy simulations (LES) of Langmuir turbulence with a submesoscale temperature front.

Use NCAR LES model to solve Craik–Leibovich equations (Moeng, 1984, McWilliams et al, 1997)

\[
\frac{\partial \rho}{\partial t} + \mathbf{u}_L \cdot \nabla \rho = \text{SGS}
\]

\[
\nabla \cdot \mathbf{u} = 0
\]

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{\omega} + f\hat{z}) \times \mathbf{u}_L = -\nabla \pi - \frac{g\rho\hat{z}}{\rho_0} + \text{SGS}
\]

Computational parameters:
Domain size: 20km x 20km x -160m
Grid points: 4096 x 4096 x 128
Resolution: 5m x 5m x -1.25m
What's plotted are surfaces of large vert. velocity, colored by temperature.
Diverse types of interaction

Slide & Movies by Peter Hamlington

Both Submeso & Langmuir-scale impacts of Stokes
FIG. 12. Potential vorticity $q$ (a,e), modified Richardson number $\phi_{Ri}$ (b,f), modified Rossby number $\phi_{Ro}$ (c,g), and instability maps (d,h) in $x$-$y$ planes (top panels) and as a function of $y$ (bottom panels) for Stokes and No Stokes cases.
Submeso-Langmuir results

- Strong interactions between small & large scales are rare in this configuration.
- Two relatively independent turbulent spectral cascades near the surface. Only submeso at depth.
- Presence of waves greatly changes small scale instability character from symmetric instability to gravitational--Stokes shear force explains this!
- Key Asymptotic divide between Submeso and Langmuir Turbulence is aspect ratio/nonhydrostatic.

Conclusions

- Climate modeling is challenging partly due to the vast and diverse scales of fluid motions.

- In the upper ocean, horizontal scales as big as basins, and as small as meters contribute non-negligibly to the air-sea exchange.

- Process models, especially those spanning a whole or multiple scales, are a powerful tool in studying these connections and improving subgrid models.

- Based on present rates of increase of computing power, we will need these subgrid models for at least another century!

Many more wave-climate effects to come... stay tuned!
Extrapolate for historical perspective:
The Golden Era of Subgrid Modeling is Now!

All papers at: fox-kemper.com/research
So, even as we begin to resolve the mesoscale...

- There are many, many processes left unresolved or partially resolved.

- **Eddy Less:** For the unresolved (no eddies), need Reynolds-Average Closures (e.g., KPP, Gent-McWilliams, Redi).

- **Eddy Rich:** eddy-permitting to resolving, need Large-Eddy-Simulation Closures (e.g., Smagorinsky).

3D Turbulence Cascade

1963: Smagorinsky Scale & Flow Aware Viscosity Scaling, So the Energy Cascade is Preserved, but order-1 gridscale Reynolds #: $Re^* = UL/\nu_*$

$E(k)$

Spectral Density of Kinetic Energy

$k^{-5/3}$

forcing range

inertial range

dissipation range

$E(k) = \begin{cases} k^{-5/3} & \text{for } k < K_F \\ \text{constant} & \text{for } K_F < k < \frac{2\pi}{\Delta x} \\ k^{-2} & \text{for } \frac{2\pi}{\Delta x} < k < K_D \\ \end{cases}$

$\nu_* = \left( \frac{\gamma_h \Delta x}{\pi} \right)^2 \sqrt{\left( \frac{\partial u_*}{\partial x} - \frac{\partial v_*}{\partial y} \right)^2 + \left( \frac{\partial u_*}{\partial y} + \frac{\partial v_*}{\partial x} \right)^2}$
2D Turbulence Differs

Re* = 1

Spectral Density of Kinetic Energy

Inverse Energy Cascade

Enstrophy Cascade

Re* = 1

Inverse Energy Cascade

Forcing

Dissipation

1996: Leith Devises Viscosity Scaling, So that the Enstrophy (vorticity^2) Cascade is Preserved

\[ \nu_* = \left( \frac{\Lambda \Delta x}{\pi} \right)^3 \left| \nabla_h \left( \frac{\partial u_*}{\partial y} - \frac{\partial v_*}{\partial x} \right) \right| \]
Some MOLES Truncation Methods In Use

2d (SWE) test

- Harmonic/Biharmonic/Numerical
  - Many. Often not scale- or flow-aware
  - Griffies & Hallberg, 2000, is one aware example
- Leith Viscosity (2d Enstrophy Scaling)
- Chen, Q., Gunzburger, M., Ringler, T., 2011
- Anticipated Potential Vorticity of Sadourny
- San, Staples, Iliescu (2011, 2013)
- Approximate Deconvolution Method

Stochastic & Statistical Parameterizations

Other session going on now in Y10

Graham & Ringler, 2013 Ocean Modelling

See also Ramachandran et al, 2013 Ocean Modelling for SMOLES
QG Turbulence: Pot’l Enstrophy cascade (potential vorticity$^2$)

Re$^*$=1

F-K & Menemenlis ’08: Revise Leith Viscosity Scaling, So that diverging, vorticity-free, modes are also damped

\[
\nu_* = \left( \frac{\Delta x}{\pi} \right)^3 \sqrt{\Lambda_h^6 |\nabla h q_2 d|^2 + \Lambda_d^6 |\nabla h (\nabla h \cdot \mathbf{u}_*)|^2}
\]

Is 2D Turbulence a good proxy for neutral flow?

For a few eddy time-scales QG & 2D AGREE (Bracco et al. '04)

Barotropic Flow--Obvious 2d analogue

Bolus Fluxes--Divergent 2d flow

Sloped, not horiz.

Surface Effects?

Yes: Nurser & Marshall, 1991 JPO
No:
Movie: S. Bachman

Potential Temperature

Day 1

In real stratified flows, things are a bit more complex than in 2d. Even more than QG...

Surface Effects may dominate...
Many observations tell us:

The spectrum of potential density and buoyancy often scales as $k^{-2}$, which isn’t too far from $k^{-5/3}$.

Figure 1: Observed spectra of mixed layer potential density variance (green), temperature contribution to potential density (blue), and temperature-density co-spectrum (red) from SeaSoar towed CTD and shipboard ADCP sections (data from Ferrari and Rudnick, 2000). A dashed line indicates $k^{-2}$ scaling.
Examples: Jan 5, 07 East of Argentina
Examples: Jan 5, 07 East of Argentina
SQG Turbulence: \textbf{Surface Buoyancy \& Velocity cascade}

\[ E(k) \]

Spectral Density of Kinetic Energy

\[ E(k) = \frac{k^{-1}}{2\pi} \]

Inverse “Energy-like” Cascade

Surface Pot’l Energy Cascade

\[ \text{Re}^* = 1 \]

\[ k^{-5/3} \]

\[ k_0 \quad k_1 \quad \frac{2\pi}{\Delta x} \quad k_D \quad k \]

forcing

dissipation

Smag-Like (Inverse):

\[ \kappa_* = \left( \frac{\Upsilon \Delta x}{\pi} \right)^{4/3} \left| \frac{1}{f} \nabla_h b \right|^{2/3} \]

Leith-Like (Direct):

\[ \kappa_* = \left( \frac{\Lambda \Delta x}{2\pi} \right)^{3/2} \left[ -\frac{\partial}{\partial z} |\nabla_h \psi|^2 \right]^{1/2} \]

W. Blumen, 1978 JAS

Held et al 1995, JFM.

Smith et al. 2002, JFM
Spectra: Jan 5, 07 East of Argentina
It is not clear that inertial ranges exist. This spectrum shows that topographic interactions change the spectrum at depth dramatically.
Reynolds vs. Péclet: Prandtl=1?

- In all cascade examples, the truncation occurs at large Reynolds and Péclet, so it is reasonable to assume diffusivity=viscosity.

- In the QG framework, diffusivity *must* equal viscosity to avoid spurious generation of potential vorticity by the subgrid model.

- For Baroclinic QG eddies, Dukowicz & Smith (97) showed that GM coefficient should equal Redi diffusivity.

- Thus, viscosity=diffusivity=GM coefficient.
And it is ... ongoing

- Scott Bachman (DAMTP) has implemented this QG Leith closure in the MITgcm.
  - Both Germano Dynamic and Fixed Coefficient.
  - Sets viscosity = diffusivity = GM coefficient.
  - Both are stable and robust.
  - Both work better than Smagorinsky, smoother spectrum to grid scale.

But, we don’t yet understand the spectral behavior of all test cases. 2d barotropic,
A Prescription for Parameterization...
Accuracy TBD

QG Leith & Potential Vorticity to generate #1 viscosity

2D Leith & Barotropic Vorticity to generate #2 viscosity

SQG Leith & Surf. Buoyancy to generate #3 diffusivity

Take max(#1, #2, #3) as viscosity, Redi diffusivity, *and* as GM transfer coeff.

Note: Unlike Eddy-Free closures, e.g., Visbeck et al (97), Eddy-Rich closures take advantage of resolved eddies & instabilities, only need a boost from eddy-permitting to eddy-resolving (and for numerical stability)

Nearly suggested by Roberts & Marshall, 98, JPO
So, no problems? Just crunch away with CLB?

Let's revisit our assumptions for scale separation:

- CLB wave equations require limited *wave steepness* and irrotational flow
- Real wind-waves are not monochromatic, but incorporate a spectrum of waves, and...

\[
\langle \eta^2 \rangle = \int_0^\infty E(k) dk = C_0 + \int_{k_n}^\infty C_1 k^{-2} dk
\]

\[
\langle k^2 \eta^2 \rangle = \int_0^\infty k^2 E(k) dk = D_0 + \int_{k_n}^\infty D_1 dk
\]

Power Spectrum of wave height

Power Spectrum of wave steepness: INFINITE!

Steep waves break -> vortex motion & small scale turbulence!
A **Global Parameterization** of **Mixed Layer Eddy Flow** & **Scale Aware Restratification** validated against simulations


\[
\mathbf{u}' b' \equiv \Psi \times \nabla \bar{b}
\]

\[
\Psi = \begin{bmatrix} \Delta x \\ L_f \end{bmatrix} \frac{C_e H^2 \mu(z)}{\sqrt{f^2 + \tau^{-2}}} \nabla \bar{b} \times \hat{z}
\]

Compare to the original **singular, unrescaled** version

\[
\Psi = \begin{bmatrix} C_e H^2 \mu(z) \end{bmatrix} \frac{1}{|f|} \nabla \bar{b} \times \hat{z}
\]

New version handles the equator, and averages over many fronts