A Perturbation Approach to Understanding the Effects of Turbulence on Frontogenesis

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Ocean fronts are an important submesoscale feature yet frontogenesis theory often neglects turbulence–even parameterized turbulence–leaving theory lacking in comparison with observations and models, especially for fronts whose development is arrested by turbulence. A perturbation analysis is used to include the effects of eddy viscosity and diffusivity as a first order correction to existing strain-induced inviscid, adiabatic frontogenesis theory. A modified solution is obtained by using potential vorticity and surface conditions to quantify turbulent fluxes. It is found that horizontal viscosity and vertical diffusivity tend to be frontolytic and oppose front sharpening, whereas vertical viscosity and horizontal diffusivity tend to be frontogenetic and strengthen the front.

Surface quasi-geostrophic theory–neglecting all injected interior potential vorticity–is able to describe the first order correction to the along-front velocity, and both surface conditions and interior potential vorticity injection are necessary to describe the correction to the ageostrophic overturning circulation. Local conditions near the front maximum are sufficient to reconstruct the along-front velocity correction everywhere.

Key words:

1. Introduction

The vast role of the ocean in the climate system spans from global processes such as water mass transport, sea level and heat content changes, to small scale processes such as mixing heat, momentum and air-sea interactions (Siedler et al. 2013). The ocean mixed layer is the upper most layer of the ocean and is in contact with the atmosphere. Due to winds and waves the ocean mixed layer is characterized by being well mixed vertically, and thus weakly stratified. Because the mixed layer is in contact with the atmosphere, transfers of heat and momentum pass through it, as well as other tracers such as carbon, to the ocean interior (Bachman et al. 2017).

The large scale flow is related to smaller scale flows by transfer of energy between the scales (Wunsch & Ferrari 2004), down to turbulence and the dissipative scales (Ferrari & Wunsch 2009; Houghton et al. 2001; Capet et al. 2008; Molemaker et al. 2010; Callies et al. 2016). Mesoscale eddies span as large as hundreds of kilometers in horizontal

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length and months in time evolution, and tend to merge, transferring kinetic energy from smaller scales to the larger scales (inverse cascade), and so do not provide an easy route to dissipation (Ferrari & Wunsch 2009), except through intermittent interactions with boundary layers (Pearson & Fox-Kemper 2018). Submesoscale currents are thought to be a key component in the forward energy cascade in the ocean (McWilliams 2016). They span the range of \(0.1 - 10\text{km}\) in horizontal scale, \(0.01 - 1\text{km}\) in vertical scale and hours to days in time evolution. Because of the fast time scale, submesoscales respond faster to atmospheric forcing and play an important role in atmosphere-ocean interactions in the mixed layer (Renault et al. 2018). For these exact reasons, it has been challenging to study submesoscale currents since they are small and impermanent requiring new strategies for ship surveys, satellite detection and global climate models. Submesoscale length and time scales, together with typical mixed layer stratification and instabilities, complicate the theoretical study of submesoscale dynamics (McWilliams 2016).

Fronts are an important and ubiquitous submesoscale feature of the mixed layer. They are characterized by elongated sharp horizontal density gradients, and an ageostrophic overturning circulation in the interior, working to restore stratification and thermal wind balance as mixing and strain alter the front. Due to the vertical properties of the ageostrophic overturning circulation, fronts are thought to play an important role in transporting tracers and supplying essential nutrients to marine biology (Olita et al. 2017; Mahadevan & Archer 2000; Mahadevan 2016; Smith et al. 2016; Taylor & Ferrari 2011). The ageostrophic overturning circulation has also recently been shown to be associated with the formation of gravity currents (Pham & Sarkar 2018). Much of the original theory of fronts was developed for the atmosphere, where they are critical for the understanding and prediction of weather.

The classic–inviscid, adiabatic–theory (Hoskins & Bretherton 1972; Hoskins 1982) of frontal evolution, also referred to as frontogenesis, predicts that the cross-frontal scale becomes infinitely thin in finite time. This unphysical outcome does not comply with observations both in the ocean and atmosphere (Pollard & Regier 1992; Bond & Fleagle 1985). Frontogenesis occurs in the ocean mixed layer where stratification may be complex, especially including the mixed layer base and connected upper pycnocline, and where the ocean surface is subject to atmospheric forcing by winds and thermal variations, waves and wave breaking, as well as incoming and outgoing radiative energy at short and long wavelengths. Thus, a variety of mixed layer instabilities or forced turbulent mixing affects fronts, many at a scale consistent with the width of observed fronts (Sullivan & McWilliams 2018). However, no scaling law or uniform understanding of how arrest happens over a variety of turbulent conditions exists, and present submesoscale parameterizations need such a scaling (Fox-Kemper et al. 2011).

As fronts involve both density and velocity gradients, there are potential roles for both turbulent momentum fluxes (usually simplified here as eddy viscosity) and turbulent heat fluxes (usually simplified here as eddy diffusivity). It is not well understood what kinds of turbulent fluxes may halt frontogenesis and at what scale, and what kinds enhance it (McWilliams et al. 2015) and at what rate. For example, vertical mixing has been shown to be important for frontal evolution (Thompson 2000; Nagai et al. 2006), whereas horizontal mixing is thought to play a role in the arrest process (Sullivan & McWilliams 2018). Boundary layer mixing, specifically vertical momentum flux, has also been shown to incite frontogenesis through a process called Turbulent Thermal Wind (TTW) (McWilliams et al. 2015).

In this study we present an analytic method based on perturbation analysis to account for modest turbulent effects on frontal evolution. While constant eddy diffusivity and viscosity are utilized as concrete examples here, the approach is easily generalized to
other more realistic turbulence closures or to Large Eddy Simulation diagnostic analysis (an ongoing analysis will be reported soon). A complementary diagnostic decomposition approach has recently been proposed (McWilliams 2017), emphasizing the attribution of causes of frontogenesis to their effects. Here an asymptotic approach is taken rather than a dynamical decomposition, illuminating different aspects of frontogenesis. Section 2 presents the analytic methods and formulation of the closed equation sets, followed by conclusions and outline for future work.

2. Methods

The mathematical inviscid, adiabatic framework of strain-induced frontogenesis (Hoskins & Bretherton 1972; Hoskins 1982; Shakespeare & Taylor 2013) describes the evolution of fronts when strained by mesoscale eddies (or synoptic weather in the atmospheric case). We follow the formulation presented in Shakespeare & Taylor (2013) (hereafter ST13), for a rotating fluid in Cartesian coordinates in an incompressible, hydrostatic, Boussinesq on an untilted \( f \)-plane. Eddy viscosity and diffusivity are added as new forcing terms, but otherwise this treatment and notation follows ST13.

The velocity and pressure terms are written as:

\[
\begin{align*}
U &= \bar{U} + u(x, z, t) = -\alpha x + u(x, z, t), \\
V &= \bar{V} + v(x, z, t) = \alpha y + v(x, z, t), \\
W &= 0 + w(x, z, t), \\
P &= \bar{P} + p(x, z, t) = \rho_0 \left[ -\frac{\alpha^2 (x^2 + y^2)}{2} + f \alpha xy \right] + p(x, z, t), \\
B &= 0 + b(x, z, t).
\end{align*}
\]

Where \( \bar{U}, \bar{V}, \bar{P} \) are the background balanced, horizontal large-scale deformation fields and the associated pressure field respectively. The reference density is \( \rho_0 \), \( f \) is the Coriolis parameter and \( u, v, w, p, b \) are the laminar frontogenetic velocity, pressure, and buoyancy fields. As appropriate for a mixed layer, upper ocean problem, no background buoyancy stratification is assumed, so the background pressure field can be thought to occur primarily by sea surface height anomalies. The laminar frontogenetic fields are assumed to be nearly independent of the along-front direction \( y \), so only \( \bar{V} \) and \( \bar{P} \) vary in \( y \). The strain rate \( \alpha \) is taken as a constant, which is impossible in a true ocean with coastlines, but is an approximation valid when \( \alpha \) represents much larger-scale features, such as mesoscale eddies, that do not vary over the confluence region of the submesoscale front in question. Note that \( y \)-invariance presumes that the laminar frontogenetic variables represent laminar perturbations to the background flow. All turbulent contributions will be assumed to be scale-separated from these laminar flows and thus treated via parameterization: here eddy viscosity and diffusivity are the explicit parameterization forms carried through the analysis, although generalizations are readily handled with the same methodology. To review, the flow is decomposed as a sum of background (capital letters), laminar frontogenetic (lower case), and turbulent (parameterized so not explicitly part of \( u, v, w, b, p \), and either frontogenetic or frontolytic) pieces.

It is useful to introduce a vector streamfunction, with vertical (geostrophic) component\(^\dagger\) divided into a background and laminar frontogenetic contribution \( \Phi = \bar{\phi} + \phi(x, z, t) \) and along-front (ageostrophic) component \( \Psi = 0 + \psi(x, z, t) \). The vertical streamfunction component is related to the pressure and horizontal geostrophic velocities by separating

\(^\dagger\) Sometimes called “velocity potential”, but better understood as the vertical component of the streamfunction.
background and laminar frontogenetic contributions,
\[
\tilde{\phi} = \frac{\bar{P}}{\rho_0 f} + \alpha^2 \left(\frac{x^2 + y^2}{2}\right) = \alpha xy, \quad \phi(x, z, t) \approx \frac{p(x, z, t)}{\rho_0 f}, \tag{2.2}
\]
\[
\frac{\partial \tilde{\phi}}{\partial y} = -\bar{U}, \quad \frac{\partial \tilde{\phi}}{\partial x} = \bar{V}, \quad \frac{\partial \phi}{\partial y} = -u_g = 0, \quad \frac{\partial \phi}{\partial x} = v_g. \tag{2.3}
\]
While the along-front, or “overturning circulation”, streamfunction component is related to the ageostrophic laminar frontogenetic velocities,
\[
\frac{\partial \psi}{\partial z} = -u_a, \quad \frac{\partial \psi}{\partial x} = w = w_a. \tag{2.4}
\]
Here the term “secondary” is avoided as a description of the overturning circulation, as it is easily confused with the order of the perturbation analysis. Likewise, the phrase “laminar frontogenetic” is preferred over the more common “perturbation” velocity because it is easily confused with terms involved in the perturbation method. As in Hoskins & Bretherton (1972) (hereafter HB72), the leading-order along-front laminar frontogenetic velocities are purely geostrophic \( [u = u_g] \), and the cross-front laminar frontogenetic velocity is a combination of geostrophic and ageostrophic components \( [u = u_g + u_a] \). The background straining velocity is given by \( \tilde{\phi} \) alone.

Similar to ST13, the governing equations for the 2D laminar frontogenetic response to the background flow can be expanded out from the definitions above, assuming hydrostatic, laminar flow. Here we include the novel addition of an eddy diffusive flux \( F(b) \) and viscous flux \( F(u), F(v) \). For now, these are written in a form amenable to accommodate most present parameterizations, including spatial variation, nonlocal fluxes, and tensor character (Large et al. 1994; Griffies 1998; Fox-Kemper et al. 2008; Bachman et al. 2015, 2017).

\[
\frac{Du}{Dt} f_v = \alpha u - \frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nabla \cdot F(u). \tag{2.5a}
\]
\[
\frac{Dv}{Dt} + f u = -\alpha v + \nabla \cdot F(v). \tag{2.5b}
\]
\[
0 = b - \frac{1}{\rho_0} \frac{\partial p}{\partial z} \tag{2.5c}
\]
\[
\frac{Db}{Dt} = \nabla \cdot F(b) \tag{2.5d}
\]
\[
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{2.5e}
\]
Where \( b \) is the buoyancy and from (2.5c) and (2.2) we have the relation:

\[
\frac{\partial \phi}{\partial z} = \frac{b}{f}. \tag{2.6}
\]

The material derivative is defined as:

\[
\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (u + \bar{U}) \frac{\partial}{\partial x} + w \frac{\partial}{\partial z} \tag{2.7}
\]
Equation (2.5e) will be automatically satisfied if the streamfunctions \( \phi \) and \( \psi \) are chosen as the prognostic variables instead of \( u, v, w \).

For concreteness, we will assume in the following that the viscous and diffusive fluxes
can be written as horizontal and vertical fluxes, assumed to be down-gradient with constant viscosities and diffusivities:

\[
F(u) = \hat{x} \nu H \frac{\partial u}{\partial x} + \hat{z} \nu \frac{\partial u}{\partial z},
\]

\[
F(v) = \hat{x} \nu H \frac{\partial v}{\partial x} + \hat{z} \nu \frac{\partial v}{\partial z},
\]

\[
F(b) = \hat{x} \kappa H + \hat{z} \kappa \nu \frac{\partial b}{\partial z}.
\]

Consistent with assuming that the laminar fields do not vary in \(y\), we assume that turbulent fluxes are statistically homogeneous in \(y\) and thus the \(\hat{y}\) component can be ignored. One might be tempted to suggest that molecular viscosity and diffusivity might be used here, but we will soon see that asymptotic ordering demands an upper and a lower limit on Reynolds and Péclet number, so keep in mind that these terms are intended as turbulence parameterizations.

Within the mixed layer, these fluxes are assumed to be a direct result of turbulence, whereas in the very near-surface boundary they can be matched to applied wind shear (\(\tau/\rho_0\)) and thermodynamic forcing (\(Q\)) via frictional and diabatic flux boundary conditions:

\[
\frac{\tau_x}{\rho_0} = \hat{z} \nu \frac{\partial u}{\partial z}
\]

\[
\frac{\tau_y}{\rho_0} = \hat{z} \nu \frac{\partial v}{\partial z}
\]

\[
Q = \hat{z} \kappa \nu \frac{\partial b}{\partial z}
\]

Penetrating solar fluxes can be accounted for, but are neglected here: all thermodynamic forcing is taken to be at the near-surface.

### 2.1. Dimensionless expressions

Following ST13, we use the following scales to make the perturbation field equations dimensionless: the horizontal and vertical buoyancy gradients (\(M^2 = \frac{\partial b}{\partial y}\) and \(N^2 = \frac{\partial b}{\partial z}\), which is also the buoyancy frequency squared), the horizontal and vertical length scales (\(L, H\)), and the strain and Coriolis rate parameters (\(\alpha, f\)). The dimensionless expressions for quantities of interest are given in table 1. As in ST13, the vertical dimensionless coordinate ranges from 0 to 1, 0 being the bottom of the mixed layer and 1 the surface. The cross-frontal coordinate is centered around the initial front maximum. We focus on the semi-geostrophic limit for the background and laminar frontogenetic velocities, which implies that the along-front velocity is purely geostrophic, and it is scaled accordingly.

The dimensionless versions of equations (2.5b) and (2.5d) are, after reorganizing:

\[
\left[ \frac{\partial}{\partial t} + (Ro u + \gamma U) \frac{\partial}{\partial x} + Ro w \frac{\partial}{\partial z} \right] v = -\frac{1}{Ro} u - \gamma \nu + \left( \frac{Ro^2}{Re_H} \right) \frac{\partial^2 v}{\partial x^2} + \left( \frac{Ro^2}{Re_V} \right) \frac{\partial^2 v}{\partial z^2}
\]

\[
\left[ \frac{\partial}{\partial t} + (Ro u + \gamma U) \frac{\partial}{\partial x} + Ro w \frac{\partial}{\partial z} \right] b = \left( \frac{Ro^2}{Re_H} \right) \frac{\partial^2 b}{\partial x^2} + \left( \frac{Ro^2}{Re_V} \right) \frac{\partial^2 b}{\partial z^2}
\]

The dimensionless relations for \(\phi, \psi, b\) and the velocities are:

\[
u = -\frac{\partial \psi}{\partial z}, \quad \frac{\partial \phi}{\partial x} = v, \quad \frac{\partial \psi}{\partial x} = w, \quad \frac{\partial \phi}{\partial z} = b.
\]
Table 1. Dimensionless expressions for quantities of interest following ST13 framework in the semi-geostrophic limit, which implies that the along-front velocity is purely geostrophic.

Based on these scalings, the error made in assuming \( v \) is geostrophic rather than using all of (2.5a) is \( \mathcal{O}(Ro) \) following the approach to the semi-geostrophic equations of Hoskins (1975). This assumption implies that if the frictional terms are to contribute significantly when compared to the neglected ageostrophic terms in (2.5a), at least one of \( Re_H, Re_V, Pe_H, \) or \( Pe_V \) should be smaller than \( Ro \) which is a small parameter. Thus, at least one dissipative term must arise as an eddy parameterization, rather than through molecular values with Reynolds and Péclet much larger than one. For the purposes of this paper, qualitative inferences of the effects of dissipation are sought, but in ongoing work diagnosing Large Eddy Simulations, a comparison of the laminar frontogenetic velocities to resolved turbulence is being evaluated directly.
Following (Hoskins 1982), we take the $z$ derivative of equation (2.12a) and the $x$ derivative of equation (2.12b), and using (2.13), reduce to one frontogenesis equation representing the instantaneous ageostrophic streamfunction governed by the strain field, geostrophic field and frictional terms:

$$
\frac{\partial^2 \phi}{\partial z^2} \frac{\partial^2 \psi}{\partial x^2} - 2 \frac{\partial^2 \phi}{\partial z \partial x} \frac{\partial^2 \psi}{\partial x \partial z} + \left( \frac{\partial^2 \phi}{\partial z^2} + \frac{1}{Ro^2} \right) \frac{\partial^2 \psi}{\partial z^2} = \\
\frac{2\gamma}{Ro} \frac{\partial^2 \phi}{\partial x \partial z} - \left( \frac{Ro}{Re_H} \right) \frac{\partial^4 \phi}{\partial x^4} - \left( \frac{Ro}{Re_V} \right) \frac{\partial^4 \phi}{\partial x^2 \partial z^2} + \left( \frac{Ro}{Pc_H} \right) \frac{\partial^4 \phi}{\partial x \partial z^3} + \left( \frac{Ro}{Pc_V} \right) \frac{\partial^4 \phi}{\partial z^4} \frac{\partial^2 \phi}{\partial x^2}.
$$

In ST13 and HB72 an analytic solution is obtained by assuming zero or constant potential vorticity (PV) everywhere in the domain. This class of solutions will be the zeroth-order starting point for the perturbation analysis here.

### 2.2. Perturbation analysis

For a small term $\varepsilon$ we use perturbation analysis to account for the effects of turbulence. We construct the zeroth-order solution to contain all the leading order laminar frontogenetic dynamics described in ST13 and HB72. Additionally, the background flow (i.e., $\bar{U}, V$) is also part of the zeroth order, so for the inviscid, adiabatic, zeroth-order limit a zero PV valid solution exists.

$$
\begin{cases}
U = \bar{U} + u = \varepsilon^0(\bar{U} + u^0) + \varepsilon^1 u^1 + O(\varepsilon^2), \\
V = \bar{V} + v = \varepsilon^0(\bar{V} + v^0) + \varepsilon^1 v^1 + O(\varepsilon^2), \\
W = w = \varepsilon^0 w^0 + \varepsilon^1 w^1 + O(\varepsilon^2), \\
\Phi = \bar{\Phi} + \phi = \varepsilon^0(\bar{\Phi} + \phi^0) + \varepsilon^1 \phi^1 + O(\varepsilon^2), \\
b = \varepsilon^0 b^0 + \varepsilon^1 b^1 + O(\varepsilon^2).
\end{cases}
$$

At the first order, the effects of turbulence (2.8)–(2.10) and the associated surface forcing (2.11a)–(2.11c) on the laminar frontogenetic flow will be isolated and examined.

We will now study individually horizontal and vertical viscosity and diffusivity. For each case the small perturbation parameter $\varepsilon$ is defined by:

$$
\begin{cases}
\varepsilon_{HV} = \frac{Ro}{Re_H} = Ek_H : \text{ Horizontal viscosity (HV)} \\
\varepsilon_{VV} = \frac{Ro}{Re_V} = Ek_V : \text{ Vertical viscosity (VV)} \\
\varepsilon_{HD} = \frac{Ro}{Pc_H} = Pr_H : \text{ Horizontal diffusivity (HD)} \\
\varepsilon_{VD} = \frac{Ro}{Pc_V} = Pr_V : \text{ Vertical diffusivity (VD)}.
\end{cases}
$$

Where $Ek_H = \frac{Ro}{Re_H} = \frac{\nu}{\mu L^2}$, $Ek_V = \frac{Ro}{Re_V} = \frac{\nu}{\mu H^2}$ and $Pr_H = \frac{\nu H}{\mu}$, $Pr_V = \frac{\nu V}{\mu}$ are the horizontal and vertical Ekman and Prandtl numbers. For small Ekman numbers and $O(1)$ Prandtl numbers, these terms are all expected to be small. However, depending on the details of the type of turbulence approximated, it may be expected that they may not be the same size. The consistency limitation of the Hoskins (1975) semi-geostrophic assumption is that at least one of them needs to be larger than $Ro$. So we proceed asymptotically by assuming they are all small and of equal order, and then after the asymptotic perturbation expansion we can choose to individually neglect them later. These small parameters appear before the terms of highest differential order—at least using the eddy viscosity and diffusivity parameterization (2.8)–(2.10)–in (2.14), there will always be some small “frictional sublayer” scale on which they are not small to satisfy the boundary conditions (2.11a)–(2.11c). A similarity solution may thus be more appropriate as an asymptotic approach in order to “magnify” the sublayer.
region and examine potentially leading order impacts outside of the sublayer. However, it is the intention here that the singular perturbation implied by neglecting these highest-derivative-order terms be equivalent to the traditional solution of the classic inviscid, adiabatic frontogenesis theory—which is by construction following this method the lowest-order solution here. Furthermore, other parameterizations of turbulence differ in differential order, e.g., a Newtonian drag as used in some eddy-damping boundary layer schemes (e.g., Parsons 1969; Fox-Kemper & Ferrari 2009) or Newtonian cooling as sometimes used in air-sea damping schemes (e.g., Dijkstra & Molemaker 1997). Hence, any frictional boundary layer will be specific to the differential form of the eddy viscosity and diffusivity. Furthermore, the true ocean boundary has a wavey-frothy-bubbley sublayer for which there exist numerical approaches but not analytic ones. It is not our intention in this study to find the true effects of specific eddy viscosity and diffusivity on frontal evolution. Rather, we use this analysis as a guide to later analyze realistic turbulent frontal simulations.

Thus, (2.14) is solved assuming an ansatz of regular perturbation analysis, and that the solution exists outside the frictional sublayer, affected only at first order by the turbulence there. We insert the above into equation (2.14) found in the previous section and separate by order of $\varepsilon$.

### 2.2.1. Zeroth order: inviscid, adiabatic

This order is the traditional frontogenesis regime, as studied by HB72 and ST13. Typically, a zero or constant potential vorticity is assumed to arrive at a simpler solution. We will preserve this assumption for zeroth order, but it will be revisited in the context of dissipative turbulence fluxes below. Additionally, to ensure consistency with semi-gesotrophy and asymptotic theory, we confine this analysis to $\gamma = O(Ro^2)$ and $Ro < \varepsilon < O(Ro^{-2})$. The zeroth order frontogenesis equation for the streamfunction, equation (2.14) equivalent to HB72, is:

$$\frac{\partial^2 \psi_0}{\partial z^2} - 2 \frac{\partial^2 \psi_0}{\partial x \partial z} + \left( \frac{\partial^2 \psi_0}{\partial x^2} + \frac{1}{Ro^2} \right) \frac{\partial^2 \psi_0}{\partial z^2} = \frac{2}{Ro^2} \frac{\partial^2 \psi_0}{\partial x \partial z}. \quad (2.17)$$

ST13 introduced a new coordinate system, similar to the geostrophic momentum coordinate in HB72:

$$X = e^\alpha t \left( x + \frac{v_0}{f} \right), \quad Z = z, \quad T = t. \quad (2.18)$$

This $X$ coordinate is conserved for any value $\alpha$ and is referred to as “generalized momentum coordinate”. A similar expression for the vertical coordinate is desirable when the scale of gradient sharpening in the vertical is comparable to the horizontal (e.g. McWilliams et al. 2009). Here we assume frontal sharpening is dominated by the horizontal strain field, and thus strictly use the horizontal form of the generalized momentum coordinate. Note that since ST13 solve the inviscid non-diffusive case, this coordinate system is associated with the zeroth order solution.

The dimensionless form of this coordinate system is:

$$X = e^{\gamma T} \left( x + Ro \ v_0 \right), \quad Z = z, \quad T = t. \quad (2.19)$$

In this new coordinate system, which tracks the Lagrangian displacements in $x$, the dimensionless material derivative reduces to

$$\frac{D}{Dt} = \frac{\partial}{\partial T} + Ro \ v_0^0 \frac{\partial}{\partial Z}. \quad (2.20)$$
And the dimensionless Jacobian for this transformation is:
\[ J = e^\gamma T \left( 1 - e^\gamma T \frac{\partial b^0}{\partial X} \right)^{-1}. \] (2.21)

The solution for all the zeroth order terms, assuming zero PV, are given in ST13. The buoyancy field is defined as:
\[ b^0(X, Z, T) = B^0_0(X) + Fr^{-2}Z + \Delta b^0(X, Z, T) \] (2.22)

Where \( B^0_0(X) \) is the initial imposed buoyancy field as a function of \( X \). In ST13 \( B^0_0(X) = \frac{1}{2} \text{erf} \left( \frac{X}{\sqrt{2}} \right) \) which has infinite derivatives. For simplicity, we take the HB72 scenario (for which \( v = v_g \)) and the corresponding solution from ST13 is the following:
\[ u^0 = -e^\gamma T \frac{\partial v^0}{\partial X} \left( Ro w + (2Z - 1)\gamma \right) \] (2.23a)
\[ w^0 = Ro \gamma B''_0(X)Z(Z - 1)e^{2\gamma T} \left( 1 - Ro e^\gamma T \frac{\partial v^0}{\partial X} \right)^{-1} \] (2.23b)
\[ v^0 = e^\gamma T \frac{\partial v^0}{\partial X} \left( Z - \frac{1}{2} \right) \] (2.23c)

Later it will be useful when using the zeroth order velocities to convert to the coordinate system (2.19) where the velocities are written explicitly and have relatively simple expressions.

The criterion in ST13 for frontal singularity is that the transformation no longer holds, i.e., the inverse of the Jacobian vanishes \( J^{-1} = e^{-\gamma T} - Ro \frac{\partial v^0}{\partial X} \). This happens when \( B'''_0(X_f) = 0 \). For the HB72 case, the singularity forms at the critical time of \( t = 19.8 \), and the Eulerian location of this singularity is found to be:
\[ x_f = Ro \frac{1}{\sqrt{2\beta}} (X_f \beta + B'_0(X_f)) \] (2.24)

Where \( \beta = -B''_0(X) > 0 \). Note that we expect \( x_f \) to appear in two locations: one at the surface, cold-side corner of the front in the mixed layer (\( x_{ft} \)) and the other at the base, warm-side corner of the front (\( x_{fb} \)).

Figure 1 illustrates the evolution of frontogenesis, as calculated by the zeroth order buoyancy field with \( Ro = 0.4, \gamma = 0.1 \). As frontogenesis progresses, the imposed strain increases the buoyancy gradient, which, through thermal wind balance, strengthens the along-front shear (figure 2). A useful diagnostic measure for frontal tendency is the Lagrangian evolution of the horizontal buoyancy gradient \( T^0_b = \frac{D}{Dt} \frac{1}{2} \left( \frac{\partial b^0}{\partial x} \right)^2 \) (Hoskins 1982; McWilliams et al. 2015), shown in figure 3. Positive frontal tendency coincides with the along-front velocity extremum, and in late frontogenesis a negative frontal tendency adjacent to the front maximum, contributes to the sharpening the front, eventually leading to singularity. An ageostrophic overturning circulation appears in the cross-frontal plane, attempting to re-stratify the front, further contributing to frontogenesis (figure 4). The overturning streamfunction in figure 4 indicates a counterclockwise overturning, which is in the direction to move buoyant water over dense and stratify the frontal region. However, as the frontal region is somewhat wider than the overturning, the upper left (upper, cold-side) and lower right (lower, warm-side) buoyancy gradients are concentrated more than other regions of the front. Illustrated in the right panels of figures 2 and 4, as frontogenesis progresses, the ageostrophic streamfunction strengthens and narrows, and the along-front velocity gets pinched to two points at the top, cold-side corner and
bottom, warm-side corner of the frontal region of the mixed layer, and the buoyancy gradient strengthens and isopycnals come closer together and tilt—especially in these two corners, consistent with the locations of maximum along-front velocity and frontal tendency. The zeroth order solution continues to strengthen and narrow, and becomes singular at these two points within a finite time. In the following section, the first order solution effects on this this process are shown.

2.2.2. First order: dissipation

For each case the first order frontogenesis equation will be slightly different, depending on the type of forcing at hand:

$$\frac{\partial^2 \phi^0}{\partial z^2} \frac{\partial^2 \psi^1}{\partial x^2} - 2 \frac{\partial^2 \phi^0}{\partial z \partial x} \frac{\partial^2 \psi^1}{\partial x \partial z} + \left( \frac{\partial^2 \phi^0}{\partial x^2} + \frac{1}{Ro^2} \right) \frac{\partial^2 \psi^1}{\partial z^2} =$$

$$- \frac{\partial^2 \phi^1}{\partial x^2} \frac{\partial^2 \psi^0}{\partial z^2} + 2 \frac{\partial^2 \phi^1}{\partial z \partial x} \frac{\partial^2 \psi^0}{\partial x \partial z} - \frac{\partial^2 \phi^1}{\partial z^2} \frac{\partial^2 \psi^0}{\partial x^2} + \frac{2 \gamma}{Ro} \frac{\partial^2 \phi^1}{\partial x \partial z} + F(\phi^0), \quad (2.25)$$
Figure 2. Cross-frontal profile of zeroth order buoyancy field at two times defined as early frontogenesis ($t = 0.9$) and late frontogenesis ($t = 11.9$), which correspond to the two sides of the box in figure 1. Superimposed in black contours is the along-front velocity. Solid lines represent positive velocity and dashed negative.

Figure 3. Zeroth order frontal tendency $T_b^0$ at two times defined as early frontogenesis ($t = 0.9$) and late frontogenesis ($t = 11.9$), which correspond to the two sides of the box in figure 1. Superimposed in black contours is the along-front velocity. Solid lines represent positive velocity and dashed negative. Note the different colorbar axis.

The different dissipative parameterizations arise as:

$$
\mathcal{F}(\phi^0) = \begin{cases} 
-\frac{\partial^4 \phi^0}{\partial x^3 \partial z} & \text{(HV)} \\
-\frac{\partial^4 \phi^0}{\partial z^3 \partial x} & \text{(VV)} \\
\frac{\partial^4 \phi^0}{\partial x^3 \partial z} & \text{(HD)} \\
\frac{\partial^4 \phi^0}{\partial z^3 \partial x} & \text{(VD)}
\end{cases}
$$

(2.26)
In our theoretical framework both surface boundary conditions and interior turbulent fluxes enter at the first order and are functions of zeroth order terms. Thus (2.25) is an inhomogeneous (with turbulent flux divergences as forcing), 2nd-order, linear (from perturbation method), partial differential equation (PDE) which is coupled for the unknown overturning and vertical streamfunction perturbations $\psi^1, \phi^1$ with non-constant coefficients, which are known functions of the zeroth order solution. The perturbation analysis is restricted to early frontogenesis development, when $\psi^1 < O(1/\epsilon)$ so perturbation analysis is appropriate and also at early times the PDE (2.25) remains elliptic.

Note that the viscous terms are of opposite sign to the diffusive terms, but horizontal diffusivity and viscosity have the same operator, as do vertical diffusivity and viscosity. Thus, **diffusivity and viscosity can not act together along the same direction to both arrest frontogenesis.** If one tends to strengthen the front, the other will resist strengthening. The turbulent Prandtl number takes on a key role, as it distinguishes a net frontogenetic dissipation from frontolytic by quantifying the relative importance of viscosity and diffusivity. An additional equation is required in order to obtain uncoupled solutions for the first order overturning and geostrophic streamfunction corrections $\phi^1, \psi^1$.

### 2.3. Potential Vorticity

Potential vorticity (PV), specifically Ertel PV, is a conserved quantity fundamental to geophysical fluid dynamics, and is useful in understanding oceanic and atmospheric dynamics (Rhines 1986; Hoskins 1991; Gill 1982; Pedlosky 1987; Salmon 1998; Kurgansky & Pisnichenko 2000). PV is defined from the absolute vorticity ($\omega = f k + \nabla \times u$) and buoyancy gradient:

$$ q = (f k + \nabla \times u) \cdot \nabla b \quad (2.27) $$

Where $u = (\bar{U} + u, \bar{V} + v, w)$ is the laminar velocity field. Note that turbulent velocities are not included in this definition of PV, which is important in the interpretation of the eddy parameterizations. Since the zeroth order PV is assumed to be zero as in traditional frontogenesis theory, any PV in the perturbation system is associated with the first order
Figure 5. Cross-frontal profile of the first order potential vorticity (coloring) at $t = 6.9$, for the four forcing cases: horizontal viscosity, vertical viscosity, horizontal diffusivity, vertical diffusivity. Positive values are given in orange and negative in purple. Notice the different colorbar range among the different forcing cases. Superimposed in black contours is the along-front velocity. Solid lines represent positive velocity and dashed negative. The PV maximum and minimum points are highlighted by light and dark green.

and is a result of turbulent fluxes and boundary injection. The dimensionless first order PV is:

$$\frac{q^1}{Ro^2} = \frac{\partial v^1}{\partial x} \frac{\partial b^0}{\partial z} - 2 \frac{\partial v^0}{\partial z} \frac{\partial b^1}{\partial x} + \left( \frac{1}{Ro^2} + \frac{\partial v^0}{\partial x} \right) \frac{\partial b^1}{\partial z}. \quad (2.28)$$

Or in terms of $\phi^1$,

$$\frac{q^1}{Ro^2} = \frac{\partial b^0}{\partial z} \frac{\partial^2 \phi^1}{\partial x^2} - 2 \frac{\partial v^0}{\partial z} \frac{\partial^2 \phi^1}{\partial x \partial z} + \left( \frac{1}{Ro^2} + \frac{\partial v^0}{\partial x} \right) \frac{\partial^2 \phi^1}{\partial z^2}. \quad (2.29)$$

The PV equation (2.29), like (2.25), is also an inhomogeneous (the forcing here is the non-zero PV), 2nd-order, linear elliptic PDE for $\phi^1$ with non-constant coefficients which are functions of the zeroth order solution. As the zeroth order solution is already known, then if given the strength of $q^1$ the problem (2.29) can be inverted to find $\phi^1$. From this, $\psi^1$ can be found using (2.25). So, if $q^1$ is known, then (2.29) and (2.25) are two coupled, linear PDEs in the unknowns $\phi^1, \psi^1$.
The evolution of PV (which is just $q^1$ as the zeroth order has zero PV) is determined by turbulent fluxes through the so-called J-equation, and can be written in terms of an advective term, a frictional flux term and diabatic flux term (Haynes & McIntyre 1987; Marshall & Nurser 1992; Thomas 2005; Benthuysen & Thomas 2012; Wenegrat et al. 2018):

$$\frac{\partial q}{\partial t} = -\mathbf{u} \cdot \nabla q + (\nabla \times \mathbf{D}_u) \cdot \nabla b + \omega \cdot (\nabla D_b)$$  \hspace{1cm} (2.30)

Where $D_u$ and $D_b$ are the frictional and diabatic flux divergences.

Since the zeroth-order PV is uniform, the total PV equation reduces to the evolution of the first order PV.

$$\frac{\partial q^1}{\partial t} = -\mathbf{u}^0 \cdot \nabla q^1 + (\nabla \times \mathbf{D}_u^1) \cdot \nabla b^0 + \omega^0 \cdot (\nabla D_b^1)$$  \hspace{1cm} (2.31)

In the asymptotic framework, the frictional and diabatic fluxes appear only in the first order and are functions only of known zeroth-order factors:

$$D_u^1 = \nabla \cdot F(v^0)$$ \hspace{1cm} (2.32a)  
$$D_b^1 = \nabla \cdot F(b^0)$$ \hspace{1cm} (2.32b)

Likewise, the advection of the first-order PV is only carried by the known zeroth-order velocity: $(\bar{U} + u^0, \bar{V} + v^0, w^0)$.

The non-dimensional form of this equation is:

$$\frac{\partial q^1}{\partial t} = - \left[ (Ro u^0 + \gamma \bar{U}) \frac{\partial}{\partial x} + Ro w^0 \frac{\partial}{\partial z} \right] q^1 + \mathcal{D}(v^0, b^0)$$  \hspace{1cm} (2.33)

Where the turbulence parameterization effects on PV have been collected into

$$\mathcal{D}(v^0, b^0) = \begin{cases} 
\frac{\partial b^0}{\partial x} \frac{\partial^2 v^0}{\partial x \partial z} - \frac{\partial v^0}{\partial z} \frac{\partial^2 b^0}{\partial x \partial z} & \text{(HV)} \\
\frac{\partial b^0}{\partial x} \frac{\partial^2 v^0}{\partial x^2} - \frac{\partial v^0}{\partial z} \frac{\partial^2 b^0}{\partial x^2} & \text{(VV)} \\
\frac{\partial v^0}{\partial z} \frac{\partial^2 b^0}{\partial z^2} - \frac{\partial b^0}{\partial z} \frac{\partial^2 v^0}{\partial z^2} & \text{(HD)} \\
\frac{\partial v^0}{\partial z} \frac{\partial^2 b^0}{\partial z \partial x} - \frac{\partial b^0}{\partial z} \frac{\partial^2 v^0}{\partial z \partial x} & \text{(VD)} 
\end{cases}$$  \hspace{1cm} (2.34)

### 2.4. First-Order Solution Procedure

To evaluate first-order solutions for each of the frictional forcing cases, equation (2.33) is integrated in time using the zeroth order terms from the previous section. During integration, the first-order PV is calculated, advected, and accumulated. The resulting first-order PV is shown in figure 5 for each case at time $t = 6.9$, exemplifying early frontogenesis. Superimposed is the along-front velocity contours for a sense of where the front is strongest, which typically correspond to the location of the PV extrema (green markers). The PV appears to be a good indicator of the center of frontogenetic activity in the Lagrangian frame, as PV primarily, and especially the extrema, are confined to the region where the front is strongest. Even under more complex scenarios of unstable frontogenesis, PV remains a good indicator of evolving mixed layer fronts, as intruding non-zero PV tends to center on the regions of frontal activity (e.g., Boccaletti et al. 2007).

The first order solution is next found, for every forcing case, by solving equations...
Figure 6. Total cross-frontal streamfunction $\psi = \psi^0 + \varepsilon \psi^1$ during early frontogenesis ($t = 6.9$) for: horizontal viscosity, vertical viscosity, horizontal diffusivity, vertical diffusivity, no forcing (zeroth order solution). Superimposed is the along-front velocity. Solid lines represent positive velocity and dashed negative. The PV maximum and minimum points are highlighted by light and dark green.

(2.25) and (2.29) using the PV resulting from integration in time of (2.33). By integrating equations (2.5a) and (2.5d), the vertical boundary conditions are solved at each time, and horizontal boundary conditions assume the laminar frontogenetic fields vanish in the horizontal, far from the front (more details in Appendix A).

2.5. First-Order Results

Results reveal the wide range of effects turbulent fluxes may have on the evolution of a front. For each forcing case, the total cross-frontal streamfunction $\psi = \psi^0 + \varepsilon \psi^1$, with $\varepsilon = 0.1$, is shown in figure 6 at time $t = 6.9$. Superimposed are the along-front velocity and the locations of the PV extrema. The overturning streamfunctions indicate the direction of fluid motion, and they are useful to understand whether in future times, the front will strengthen or diffuse. The first order frontal tendency (figure 7) complements this
Figure 7. Total cross-frontal frontogenetic tendency \( T_b = T_b^0 + \varepsilon T_b^1 \) during early frontogenesis \((t = 6.9)\) for: horizontal viscosity, vertical viscosity, horizontal diffusivity, vertical diffusivity, no forcing (zeroth order solution). Superimposed is the along-front velocity. Solid lines represent positive velocity and dashed negative. The PV maximum and minimum points are highlighted by light and dark green.

Interpretation of first order effects, and is evaluated by:

\[
T_b^1 = \frac{D}{Dt_0} \left( \frac{\partial \theta^0}{\partial x} \frac{\partial b^1}{\partial x} \right) + \frac{D}{Dt_1} \frac{1}{2} \left( \frac{\partial \theta^0}{\partial x} \right)^2
\]

Where subscripts on the Lagrangian derivative operator indicate the order of the advecting velocities.

In the zeroth-order solution illustrated in figures 3 and 4, the overturning streamfunction tended to focus the buoyancy gradients in two corner points, consistent with the locations of maximum along-front velocity, and where eventually frontal singularities first develop. In the perturbation solutions combined to first order in figures 6 and 7, the behavior at these points is different depending on the which frictional forcing case is examined.
(i) Horizontal viscosity: a negative streamfunction appears near the surface, directing fluid away from the region of maximum frontogenesis. Furthermore, frontal tendency is maximum in opposite configuration with respect to the front maximum, when compared with the zeroth order solution. This indicates that the contribution of horizontal viscosity at the first order tends to reduce the ever-strengthening tendency of the zeroth-order solution.

(ii) Vertical viscosity: a negative streamfunction region occurs again near the surface, but in an opposite configuration with respect to the front maximum when compared to the horizontal viscosity case, thus directing fluid and gradients to concentrate toward the region of maximum frontogenesis, highlighted also by maximum frontal tendency. This flow indicates that the vertical viscosity at the first order tends to enhance frontogenesis, consistent with results from previous studies.

(iii) Horizontal diffusivity: the shape of the streamfunction is similar to that of the zeroth order at more advanced stages of frontogenesis, with added side-lobes away from the frontal extrema. Additionally, frontal tendency is positive near the front maximum and negative in the adjacent region, similar to the zeroth order frontal tendency during late frontogenesis. Thus, horizontal diffusivity tends to accelerate frontogenesis.

(iv) Vertical diffusivity: the streamfunction tilts slightly away from the front maximum, and at later times a negative circulation appears near the surface, diffusing the front similarly to the horizontal viscosity case. Unlike horizontal viscosity, frontal tendency is positive in the regions of maximum frontogenesis, but weaker than the zeroth order. This indicates that vertical diffusivity tends to decelerate frontogenesis, however on a longer timescale than horizontal viscosity.

These results are consistent with the earlier rough predictions based on the signs of the forcing operators in the first order streamfunction equations in section 2.2.2. The solutions in figure 6 are surprisingly complex in comparison to the simple reversal of signs in (2.34)–the solutions are not just mirror images because the surface boundary conditions for momentum and buoyancy are not simply related even though the field equations are. The boundary condition effects are isolated in the next subsection.

The first-order solution is distinguished from the zeroth-order solution in two ways, by the computed interior PV and through the altered boundary conditions that arise at first order. It is now natural to ask whether the solution is fully dependent on both or whether either the interior PV or the surface conditions dominates over the other. In the subsequent sections, we focus on the case of parameterized horizontal viscosity, closely linked to horizontal shear instability, and has been shown to be a prime contributor to frontogenetic arrest (Sullivan & McWilliams 2018).

2.5.1. Surface Quasi-Geostrophy versus Interior Quasi-Geostrophy

In the mixed layer, surface quasi-geostrophic (SQG) dynamics sometimes dominate over interior quasi-geostrophic (IQG) dynamics forced by anomalies in ocean interior PV, also called Charney (1971) QG dynamics (Bretherton 1966; Capet et al. 2008; LaCasce & Mahadevan 2006; Stamper et al. 2018).

In an SQG system, surface buoyancy anomalies are used to generate an active boundary condition, while the interior PV is taken to be uniform and inert (Blumen 1978). The full flow field is obtained by using the PV invertibility principle (Hoskins et al. 1985), with the active buoyancy field providing the boundary conditions for the inversion. In the IQG system, the boundary conditions are simplified by neglecting buoyancy anomalies and flow dynamics are assumed to be driven solely by interior quasi-geostrophic PV anomalies (QGPV). Two interesting byproducts of the perturbation analysis framework are that the first order Ertel PV, as ordinarily used in traditional frontogenesis literature,
is locally equivalent to QGPV at that order, and that the impact of the first-order solution can be uniquely and completely decomposed into SQG effects and IQG effects (figure 8). Turbulent closures affect surface buoyancy quite differently from interior PV, so: are the effects on frontal evolution by interior PV comparable to surface turbulent fluxes?

Following Bretherton (1966), the PV in (2.29) together with the buoyancy boundary condition on $\phi$ in (2.13) can thus be built from a combination of the interior QGPV, which in isolation drives the IQG system, and added delta functions based on surface buoyancy to replace the buoyancy boundary condition (Bretherton 1966), which in isolation drive the SQG system (more details in Appendix A). This is explicitly calculated by following the evolution of the first order buoyancy and cross-front velocity at the surface, for each forcing case:

$$b^1 = \frac{\partial \phi^1}{\partial z}, \quad b^1 = \int_0^t \left[ -u^0 \cdot \nabla b^1 - u^1 \cdot \nabla b^0 + \nabla \cdot F(b^0) \right] \delta(z - 0) \, dt \quad (2.36a)$$

$$u^1 = -\frac{\partial \psi^1}{\partial z}, \quad u^1 = \int_0^t \left[ -u^0 \cdot \nabla u^1 - u^1 \cdot \nabla u^0 + \gamma u^1 + \nabla \cdot F(u^0) \right] \delta(z - 0) \, dt \quad (2.36b)$$

A delta function for the bottom boundary condition is used as well, although this has little effect on the end result. It is important to note that the fluxes in the SQG system are surface turbulent fluxes associated with frontogenetic evolution. At this time we do not consider external surface forcing, such as wind stress, which has been shown to affect interior PV (Thomas & Ferrari 2008).

Figure 8 shows the first order along-front velocity $\mathcal{v}^1$ for the horizontal viscosity case, calculated from the SQG system, the IQG system, their sum and the complete system. Since we are solving a linear PDE in $\phi^1$, we expect the sum of the SQG and IQG systems to be similar to that of the complete system. The complete system shows a negative along-front velocity, centered in the surface front maximum, decreasing with depth. The IQG system shows a significantly weaker velocity, with a dipole shape about the front maximum. The SQG system appears to capture most elements of the complete solution, however the appropriate symmetry about the front axis is obtained when adding the results from the IQG system.

Unlike the equation for $\phi^1$, the streamfunction has a forcing term that cannot be easily attributed to SQG or IQG dynamics, as it is governed by zeroth order terms. For illustration purposes, we divide this forcing term into surface and interior domains (more details in Appendix A) and solve for the SQG and IQG systems separately. Figure 9 shows the first order cross-frontal streamfunction $\mathcal{v}^1$ also for the horizontal viscosity case, calculated from the SQG system, the IQG system, their sum and the complete system. Both the SQG and IQG systems result in streamfunctions with comparable magnitudes. The SQG system resembles the complete system near the surface whereas the IQG system resembles the complete system in the interior. The sum of the SQG and IQG systems is very similar to the complete system, and the differences may be attributed to our of choice of domains for each system.

Given turbulent fluxes, the first order correction of the along-front velocity and geostrophic potential can be found to a good approximation merely from surface conditions using SQG theory in this classic strain-induced frontogenesis case. This implies that parameterizations that affect the surface buoyancy will have a larger impact on frontal structure than those that affect interior PV, or the rate of PV injection.
Figure 8. First order along-front velocity $\epsilon \psi_1$ in the early frontogenesis time ($t = 6.9$) for horizontal viscosity forcing: complete solution, interior QG (IQG), surface QG (SQG), sum of IQG and SQG solutions.

2.6. Point Source Surface Conditions

In the SQG framework, we followed Bretherton (1966) and used delta functions in $z$ to replace the surface and bottom buoyancy boundary conditions. The success of this approach in capturing much of the full frontogenesis impacts on along-front velocity highlights the importance of surface buoyancy gradients. We now examine if a simplification of buoyancy in the horizontal direction as well captures much of the effect. During frontogenesis, the buoyancy has a sharp and sharpening gradient over the frontal domain. As the buoyancy singularity is approached, the cross-frontal buoyancy gradient $\frac{\partial b}{\partial x}$ may be approximated by a delta function in the cross-frontal direction, corresponding to the location of maximum frontogenesis, and buoyancy gradients elsewhere being neglected. Using this assumption, a simplified boundary condition for the geostrophic
Figure 9. First order cross-frontal streamfunction $\varepsilon \psi_1$ in the early frontogenesis time ($t = 6.9$) for horizontal viscosity forcing: complete solution, interior QG (IQG), surface QG (SQG), sum of IQG and SQG solutions.

The cross-frontal points where the buoyancy gradient is localized are denoted by $x_{fs}, x_{fb}$, which represent the location of the front nose at the surface ($fs$) and bottom ($fb$) of the mixed layer respectively.

A similar local approach can be applied to the IQG system. The PV in section 2.3 is found to be confined to the regions of maximum frontogenesis. For the IQG case we represent the interior PV as a delta function located where the front is strongest in the interior, which is just below the surface.

Potential $\phi^1$ is a Heaviside function in $x$ and a delta function in $z$:

$$\phi^1(z = 0) = \begin{cases} \psi_{\text{max}}^1 \delta(z - 0) & x > x_{fs} \\ \psi_{\text{min}}^1 \delta(z - 0) & x < x_{fs} \end{cases} \quad (2.37a)$$

$$\phi^1(z = H) = \begin{cases} \psi_{\text{max}}^1 \delta(z - H) & x > x_{fb} \\ \psi_{\text{min}}^1 \delta(z - H) & x < x_{fb} \end{cases} \quad (2.37b)$$

The cross-frontal points where the buoyancy gradient is localized are denoted by $x_{fs}, x_{fb}$, which represent the location of the front nose at the surface ($fs$) and bottom ($fb$) of the mixed layer respectively.
The motivation for considering such simplifications is Green’s functions theory, as a full flow field solution exists for $\phi^1$ and $\psi^1$ even if only a point source buoyancy gradient or PV is prescribed. An example analytic procedure for finding the first-order solution to (2.25), using delta function approximations and Green’s function theory, is given in Appendix B.

Figures 10 and 11 show results of the SQG and IQG systems with point source surface conditions and interior PV. The along-front velocity and streamfunction are very similar to the full SQG system solution. However, the along front velocity in the point source IQG system is substantially different than in the full forcing IQG result, both in magnitude and shape (upper right of figure 10 versus figure 8). The streamfunction, however, in the IQG point source system is almost identical to the full IQG solution, despite the small contribution from $\phi^1$ (upper right of figure 11 versus figure 9).

In conclusion, the IQG streamfunction is primarily forced by the zeroth order solution, and less so by the first-order geostrophic potential—that is, by the injection of frictional and diffusive PV, at least in the weak turbulence limit examined in the perturbation approach. The SQG geostrophic potential, which contributes most of the along-front velocity, can be reconstructed by surface conditions highly localized near the front maximum. From a parameterization or numerics development perspective, it is the method of regularizing—or keeping finite—the buoyancy gradient very near the maximum gradient surface expression of the front that will dominate the fidelity of a strain-induced frontal simulation, rather than turbulent fluxes elsewhere or the injection of PV.

3. Summary and Discussion

The classic problem of strain-induced frontogenesis theory arrives at a singularity within a finite time. The system described by this theory may capture the leading order dynamics, but it is shown here that turbulent fluxes are likely a key secondary component missing in aligning theory with observations and model simulations. Here the effects of turbulent surface and interior fluxes of buoyancy and PV, along with their boundary conditions, were isolated and examined using a perturbation approach.

In this study an asymptotic approach estimates the leading correction of turbulent fluxes to the current theory for strain-induced frontal dynamics. The uncoupled first-order solutions for the along-front geostrophic velocity and ageostrophic streamfunction are obtain by inverting the modified semi-geostrophic frontogenesis equation together with the first nonvanishing-order PV equation. By differentiating between horizontal and vertical eddy viscosity and diffusivity which, in reality, might be associated with parameterizations of mixed layer instabilities or boundary layer turbulence or numerical artifacts as simulated fronts approach singularity, an early understanding of the different impacts of these dissipative operators on fronts is gained. Horizontal viscosity and vertical diffusivity potentially act to weaken the front, and thus are key to understanding frontal arrest, consistent with Sullivan & McWilliams (2018). Vertical viscosity and horizontal diffusivity, by contrast, act to strengthen the front, consistent with Turbulent Thermal Wind theory (McWilliams et al. 2015; McWilliams 2017; Crowe & Taylor 2018).

As the full first-order solution depends on both turbulent boundary conditions and injected interior PV, a decomposition of the results into surface QG and interior QG subsystems isolates the contributions to the full solution. For the horizontal viscosity case, SQG dynamics are able to capture most of the along-front features, whereas IQG dynamics has a small contribution. The ageostrophic overturning circulation is only fully reconstructed by including both SQG and IQG systems. Due to the nature of frontal dynamics, features tend to be localized where the front is strongest. Considering only a
point source of buoyancy gradient in the surface boundary conditions and a point source of PV in the interior, a good approximation to the full IQG plus SQG response was obtained. However, the point source approach worked best in reproducing the along-front velocity perturbation in the SQG system, and so a full along-front velocity reconstruction hinged on the IQG response being weak. For the overturning streamfunction perturbation, both the SQG and IQG subsystems are well-approximated in point source solutions. This result implies that the IQG subsystem is primarily driven by the zeroth-order solution, rather than the first order geostrophic potential. An examination of the other turbulence parameterizations—horizontal diffusivity, vertical viscosity, and vertical diffusivity—yields similarly that highly localized SQG results dominate the along-front velocity effects while SQG and IQG effects combine to set the overturning streamfunction, but that IQG depends more on the zeroth-order solution than the frictional injection of PV. For determining the ageostrophic overturning circulation, where the frictional surface conditions come into play in addition to the buoyancy conditions, both IQG and SQG effects are equally important, implying that surface conditions and PV must not be neglected for a complete understanding of turbulent frontogenesis. In a recent paper,
Figure 11. Same as in figure 9 for the point source forcing solutions to SQG and IQG.

Wenegrat et al. (2018b) show that boundary layer turbulence can generate a source of PV at the surface, similar in magnitude to PV fluxes from wind and surface buoyancy fluxes (Thomas 2005). It is presently unclear whether our conclusions for the SQG and IQG systems hold true in the presence of strong surface forcing such as downfront wind forcing, or realistic boundary layer turbulence. However, the same perturbation methods can be used while including wind forcing in (2.13), and many of the LES being analyzed include winds.

The asymptotic method presented in this study is derived from the semi-geostrophic equations presented in HB72, implying that the Rossby number is smaller than 1. However, in the submesoscale regime, both inertial and rotational forces are important and so the Rossby number is $Ro \sim 1$ (McWilliams 2016). Furthermore, semi-geostrophic theory has been shown to be inconsistent with $Ro \sim 1$ (Barkan et al. 2019) and curved fronts (Gent et al. 1994). The modified theory presented here merely acts as a framework to study the potential effects turbulent fluxes may have on frontal evolution, and should be applied in the early frontogenesis stages, while the semi-geostrophic assumptions still hold. The expansion parameter $\varepsilon$ is taken to be constant, however the realistic
form is highly uncertain as it represents parameterized effects of turbulent fluxes in the submesoscale regime.

Submesoscale turbulence is affected largely from the dynamical interplay between vertical mixing due to atmospheric fluxes, and increase in stratification due to horizontal density gradients (Tandon & Garrett 1994; Hosegood et al. 2006; Bachman et al. 2017; Thomas et al. 2008; Callies et al. 2016; Mahadevan et al. 2010; Thomas & Ferrari 2008; McWilliams 2010). There are several flavors of submesoscale instabilities that populate the upper ocean, due to the effects of winds, waves and stratification, that probably affect the evolution of fronts: gravitational instability (Haine & Marshall 1998); symmetric instability, which is notable because it mixes PV, momentum, and buoyancy differently and vertically in the presence of down-front winds and negative PV (Hoskins 1974; Thomas & Taylor 2010; Thomas et al. 2013; Haney et al. 2015; Bachman et al. 2017); mixed layer baroclinic instability, which acts to restratify the mixed layer by slumping the gradients from horizontal to vertical (Bishop 1993; Spall 1997; Boccaletti et al. 2007; McWilliams et al. 2009; McWilliams & Molemaker 2011) and thus would enter this asymptotic approach as an overturning streamfunction rather than a diffusivity or viscosity (Fox-Kemper et al. 2008); Langmuir turbulence, which creates convergence zones at the ocean surface and reduce vertical stratification in the upper ocean. (McWilliams et al. 1997; Hamlington et al. 2014; McWilliams et al. 2015; Van Roekel et al. 2012; Garrett & Loder 1981; Skyllingstad & Samelson 2012; Suzuki et al. 2016); boundary layer turbulence, which provides a turbulent mechanism for the onset of frontal evolution merely by turbulent vertical momentum mixing (TTW) (McWilliams et al. 2015); horizontal shear instability due to the sharpening front itself (Sullivan & McWilliams 2018); and other mixing, wave breaking, and topographic effects (Garrett & Loder 1981; Thompson 2000; Teixeira & Belcher 2002; Nagai et al. 2006; Sullivan et al. 2007; Gula et al. 2016; Wenegrat et al. 2018a). Some of these phenomena require horizontal gradients such as a front to exist (e.g., symmetric instability or baroclinic instability), while others can be concentrated at fronts due to concentrated shear (e.g., horizontal shear instability), while others are ubiquitous and related generally to the presence of the ocean mixed layer and so depend more on the surface forcing than the presence of fronts (e.g., Langmuir turbulence, boundary layer turbulence).

It is intended that this framework be utilized in concert with simulations with more realistic turbulence parameterizations or resolved turbulence to highlight important regions or aspects of such simulations. By comparison to submesoscale simulations where the asymptotic limitations are exceeded ($Ro \gg 1$), it will be interesting to see the quantitative and perhaps qualitative changes that distinguish submesoscale front-turbulence interactions from those of mesoscale fronts. At present, Large Eddy Simulations (LES) (Moeng 1984; McWilliams et al. 1997) are able to resolve the submesoscale dynamics and boundary layer turbulence that motivates this theory. An examination of simulations similar to those studied in Hamlington et al. (2014); Smith et al. (2016); Suzuki et al. (2016) but including strain-inducing eddies are ongoing. In all runs, a confluent region produces strong frontogenesis, but the cross-frontal scale halts at a finite width. Motivated by this study, these runs are being analyzed. The eventual target is finding quantitative predictions for frontal width, strength, and persistence in the presence of realistic turbulent fluxes.

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Appendix A. Numerical Scheme

A.1. Parameters

- Dimensionless time \((t)\) range is \([0, 19.8]\) where the zeroth order vanishes according to ST13.
- Dimensionless cross-frontal \((y)\) range is \([-2.5, 2.5]\).
- Dimensionless vertical \((z)\) range is \([0, 1]\).
- \(\gamma = 0.1\).
- Rossby number \(Ro = 0.4\).
- The expansion parameter is given an arbitrary small value of \(\varepsilon = 0.1\) for all forcing cases.
- The Froud number is calculated by \(Fr = \frac{Ro}{Bu}\) where \(Bu = 0.1\) following ST13.

A.2. First Order Solution

- The zeroth order solution is calculated with equations (2.22) and (2.23a)-(2.23c), which serve as inputs for the consecutive steps.
- The PV is calculated by integrating the PV evolution equation (2.33) using Forward Euler method for the first three time steps after which the 3rd order Adam-Bashforth explicit method is used. The zeroth order solution is used to calculate the advection and flux terms at every time step.
- In the SQG simulation the PV is taken to be zero.
- The five point stencil method is used to invert, at every time step, the second order partial differential equation for \(\phi^1\) (2.29), using inputs from the zeroth order solution and the PV.
- A similar method is used for \(\psi\) (2.25), using inputs from the zeroth order solution and \(\phi^1\).

A.3. Boundary conditions

For the simulations presented in this paper we do not include fluxes induced by wind shear and thermodynamic forcing.

The boundary conditions for \(\psi^1\) is given by the integrating the cross-front momentum equation (2.5a). We use Forward Euler method for the first three time steps after which the 3rd order Adam-Bashforth explicit method is used. The streamfunction is evaluated at every time step by (2.4). The zeroth order solution is used as input for the zeroth order advection and the total flux terms. The first order advection terms are evaluated at every time step with the explicit method.

The boundary conditions for \(\phi^1\) is given by the integrating the thermodynamic equation (2.5d) by the same method as for \(\psi^1\), where the geostrophic potential is evaluated at every time step by (2.6).

In the IQG simulation the boundary conditions for \(\phi^1\) are taken to be trivial \(\frac{\partial \phi^1}{\partial z} = 0\).
In both the IQG and SQG systems we maintain the boundary conditions for \( \psi^1 \) since this is the equation for the ageostrophic streamfunction and is independent at each time of the quasi-geostrophic PV inversion. At past times, the Lagrangian history of the boundary conditions does affect \( \psi_1, \phi_1 \), but these historical effects do not directly accumulate in the first-order ageostrophic overturning circulation, which in the perturbation asymptotics, is diagnostically calculated from the zeroth-order fields through (2.25). The nature of this decomposition is highlighted by the characteristics of the boundary conditions in the SQG system, and equation (2.29) for the IQG system, independent of the first order Lagrangian derivative. The turbulent flux forcing term \( F(\phi^0) \) in equation (2.25) is divided into surface and interior domains for the SQG and QG case respectively.

In the point source case, the PV extrema locations are taken from the IQG case. In the SQG case, the boundary conditions are calculated by (2.37) where \( x_{ft}, \, x_{fb} \) are the points of maximum along-front velocity on the boundary.

### Appendix B. Analytic Solution for the Delta Function Approximation

To obtain an analytic framework, we chose the locations for which the zeroth order solution goes singular, as the points of maximum frontogenesis. In this approximation, the point sources of PV \((\bar{q}_{fs}, \bar{q}_{fb})\) can be found by evaluating the PV from equation (2.33) at these location.

With the assumption of point sources of PV, \( \phi^1 \) can be found using the Green’s function for the PDE (2.29), then equation (2.25) determines the first order geostrophic streamfunction \( \psi^1 \).

If the PDE is elliptic then we can perform a change of variables so that it becomes \( \nabla^2 \) in some other coordinate system. We are looking for a change of variables (\( \xi(x, z), \eta(x, z) \)) such that \( J = \xi_x \eta_z - \xi_z \eta_x \neq 0 \) for any \( x, z \) in the domain. For an equation of the sort:

\[
\mathcal{L}[\phi^1] = A\phi^1_{xx} + 2B\phi^1_{xz} + C\phi^1_{zz} = D
\]

Where:

\[
\begin{align*}
A &= Ro^2 \frac{\partial^2 \phi^0}{\partial x^2} \\
B &= -Ro^2 \frac{\partial^2 \phi^0}{\partial x \partial z} \\
C &= \left( 1 + Ro^2 \frac{\partial^2 \phi^0}{\partial x^2} \right) \\
D &= \bar{q}_{ft}\delta(x - x_{ft})\delta(z - z_{ft}) + \bar{q}_{fb}\delta(x - x_{fb})\delta(z - z_{fb})
\end{align*}
\]

Assuming \( A, B, C \) are real analytic functions in the domain, and that \( A(x, y) \neq 0 \) (i.e. \( \frac{\partial \phi^0}{\partial z} \neq 0 \)), a transformation \( \Pi(\xi, \eta) = \phi^1(x(\xi, \eta), y(\xi, \eta)) \) can be found such that:

\[
A\Pi_{\xi \xi} + 2B\Pi_{\xi \eta} + C\Pi_{\eta \eta} = D
\]

And:

\[
\begin{align*}
A &= A\xi_x^2 + 2B\xi_x \xi_z + C\xi_z^2 \\
B &= A\xi_x \eta_x + B(\xi_x \eta_z + \xi_z \eta_x) + C\xi_z \eta_z \\
C &= A\eta_x^2 + 2B\eta_x \eta_z + C\eta_z^2
\end{align*}
\]

Since our operator is elliptic we are looking for \( \xi, \eta \) that satisfy \( A = C \) and \( B = 0 \):

\[
\begin{align*}
A\xi_x^2 + 2B\xi_x \xi_z + C\xi_z^2 &= A\eta_x^2 + 2B\eta_x \eta_z + C\eta_z^2 \\
A\xi_x \eta_x + B(\xi_x \eta_z + \xi_z \eta_x) + C\xi_z \eta_z &= 0
\end{align*}
\]
Which can be found by defining the variable \( \lambda = \xi + i \eta \), reducing to one equation:

\[
A \lambda_x + \left( B \pm i \sqrt{AC - B^2} \right) \lambda_z = 0
\]  
(B 4)

\( \lambda \) is constant on the characteristics (defined on the complex plane):

\[
\frac{dz}{dx} = \frac{B \pm i \sqrt{AC - B^2}}{A}
\]  
(B 5)

Inserting the non-dimensional definitions for \( A, B, C \) gives:

\[
\frac{dz}{dx} = - \frac{\partial \phi}{\partial x} \pm i \sqrt{\left( Ro^2 \frac{\partial \phi}{\partial z} \right) \cdot (1 + Ro^2 \frac{\partial \phi}{\partial x}) - \left( Ro^2 \frac{\partial \phi}{\partial x} \right)^2}
\]  
(B 6)

Integrating the equation with respect to \( x \):

\[
\left[ z + \int \left( \frac{\partial \phi}{\partial x} / \frac{\partial \phi}{\partial z} \right) \, dx \right] \pm i \int \frac{\sqrt{\left( Ro^2 \frac{\partial \phi}{\partial z} \right) \cdot (1 + Ro^2 \frac{\partial \phi}{\partial x}) - \left( Ro^2 \frac{\partial \phi}{\partial x} \right)^2}}{\left( Ro^2 \frac{\partial \phi}{\partial x} \right)} \, dx = \text{const}
\]  
(B 7)

This is the characteristic solution for which \( \lambda \) is constant and \( \xi = Re[\lambda], \eta = Im[\lambda] \):

\[
\xi = z + \int \left( \frac{\partial \phi}{\partial x} / \frac{\partial \phi}{\partial z} \right) \, dx
\]  
(B 8a)

\[
\eta = \pm \int \frac{\sqrt{\left( Ro^2 \frac{\partial \phi}{\partial z} \right) \cdot (1 + Ro^2 \frac{\partial \phi}{\partial x}) - \left( Ro^2 \frac{\partial \phi}{\partial x} \right)^2}}{\left( Ro^2 \frac{\partial \phi}{\partial x} \right)} \, dx
\]  
(B 8b)

The problem now reduces to solving the equation in \( \eta, \zeta \) coordinate system, which is a Laplacian operator equal to an equivalent delta function:

\[
\Pi \xi \xi + \Pi \eta \eta = \frac{\mathcal{D}(\zeta, \eta)}{A(\zeta, \eta)} \sim \delta(\zeta, \eta)
\]  
(B 9)

Using the definition for \( \xi, \eta \) we can find the boundary conditions on \( \Pi \) from the boundary conditions of \( \phi^1 \).

Since \( \phi^1 \) is confined to the frontal region, it can be assumed that far from the front as \( x \to \pm \infty \) the first order velocities vanish:

\[
v^1 = \frac{\partial \phi^1}{\partial x} = \frac{\partial \Pi}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} = - \frac{\partial \Pi}{\partial \xi} \cdot \left( \frac{1}{Ro \frac{\partial \phi^0}{\partial z}} \right) = - \frac{\partial \Pi}{\partial \xi} \cdot \left( \frac{2}{e^{\gamma T} Ro^2 B_0(X)} \right)
\]  
(B 10)

Where we use the definition of the slope in \( X \) coordinates, as in ST13. The initial buoyancy in \( X \) coordinates is \( B_0(X) = \frac{2}{\sqrt{2\pi}} erf(X/\sqrt{2}) \to B'_0(X) = \frac{1}{\sqrt{2\pi}} e^{-X^2/2} \)

\[
\lim_{x \to \pm \infty} \frac{\partial \Pi}{\partial \xi} = \lim_{x \to \pm \infty} \left[ v^1 \left/ \frac{2\sqrt{2\pi} e^{X^2/2}}{e^{\gamma T} Ro^2} \right. \right] = 0
\]  
(B 11)

Also it can be shown that \( \xi \to \pm \infty \) as \( X \to \pm \infty \) and \( X \to \pm \infty \) as \( x \to \pm \infty \). Thus we now have the boundary conditions:

\[
\lim_{\xi \to \pm \infty} \frac{\partial \Pi}{\partial \xi} = 0
\]  
(B 12)
In a QG system, we use trivial boundary conditions $\frac{\partial^{2} \phi}{\partial z^{2}} = \frac{\partial^{2} \phi}{\partial z^{2}} = 0$.

$$\frac{\partial^{2} \phi}{\partial z} = \frac{\partial^{2} \phi}{\partial z^{2}} \rightarrow \frac{\partial^{2} \phi}{\partial z^{2}} = \frac{\partial^{2} \phi}{\partial z^{2}} \left( \frac{\partial^{2} \phi}{\partial z^{2}} \right) = \frac{\partial^{2} \phi}{\partial z^{2}} \left( \frac{\partial^{2} \phi}{\partial z^{2}} \right) = \frac{\partial^{2} \phi}{\partial z^{2}} \left( \frac{\partial^{2} \phi}{\partial z^{2}} \right) = \frac{\partial^{2} \phi}{\partial z^{2}} \left( \frac{\partial^{2} \phi}{\partial z^{2}} \right)$$  \hspace{1cm} \text{(B 13)}$$

We are interested in shown that at the boundaries $\frac{\partial^{2} \phi}{\partial z^{2}} \left( \frac{\partial^{2} \phi}{\partial z^{2}} \right) < \frac{\partial^{2} \phi}{\partial z^{2}} \left( \frac{\partial^{2} \phi}{\partial z^{2}} \right)$ because then:

$$0 = \frac{\partial^{2} \phi}{\partial z^{2}} \bigg|_{b.c} = \frac{\partial^{2} \phi}{\partial z^{2}} \left( \frac{\partial^{2} \phi}{\partial z^{2}} \right) < \frac{\partial^{2} \phi}{\partial z^{2}} \left( \frac{\partial^{2} \phi}{\partial z^{2}} \right) \approx \frac{\partial^{2} \phi}{\partial z^{2}} \left( \frac{\partial^{2} \phi}{\partial z^{2}} \right) \bigg|_{b.c} \rightarrow \frac{\partial^{2} \phi}{\partial z^{2}} \bigg|_{b.c} = 0 \hspace{1cm} \text{(B 14)}$$

In order to show this we can show that $\left( \frac{\partial^{2} \phi}{\partial z^{2}} \right)^{2} < \frac{\partial^{2} \phi}{\partial z^{2}}$ and $\frac{\partial^{2} \phi}{\partial z^{2}} < \frac{\partial^{2} \phi}{\partial z^{2}}$. This first we will show by using the scaling of the problem:

$$\eta = \sqrt{ \frac{A[C] - B^{2}}{A} } \cdot L = \sqrt{ \frac{L^{2} \rho}{M^{2} + L H} } \cdot L = \rho^{3} \cdot H \hspace{1cm} \text{(B 15)}$$

$$\rightarrow \left[ \left( \frac{\partial^{2} \phi}{\partial z^{2}} \right)^{2} \right] = \frac{\sqrt{A[C] - B^{2}}}{A} \cdot \frac{L^{2}}{H^{2}} \cdot L = \sqrt{ \frac{A[C] - B^{2}}{A} } \cdot L \hspace{1cm} \text{(B 16)}$$

For the HB72 case $\rho \sim \mathcal{O}(0.4)$ and $H \sim \mathcal{O}(10)$ which means that $\rho^{3} \cdot H \sim 10^{-3} \cdot 10^{1} = 10^{-2} < \eta$. Thus $\left( \frac{\partial^{2} \phi}{\partial z^{2}} \right)^{2} < \frac{\partial^{2} \phi}{\partial z^{2}}$.

Next, we want to show that $\frac{\partial^{2} \phi}{\partial z^{2}} < \frac{\partial^{2} \phi}{\partial z^{2}}$:

$$\frac{\partial^{2} \phi}{\partial x} = \left( \frac{\partial^{2} \phi}{\partial x} \right) = \left( \frac{\partial^{2} \phi}{\partial x} \right) = \left( \frac{\partial^{2} \phi}{\partial x} \right) = \left( \frac{\partial^{2} \phi}{\partial x} \right)$$  \hspace{1cm} \text{(B 17)}$$

In addition, in the same way we got equation B 13 with $\eta$, we can do the same with $\xi$:

$$0 = \frac{\partial^{2} \phi}{\partial z^{2}} \bigg|_{b.c} = \frac{\partial^{2} \phi}{\partial z^{2}} \left( \frac{\partial^{2} \phi}{\partial z^{2}} \right) < \frac{\partial^{2} \phi}{\partial z^{2}} \left( \frac{\partial^{2} \phi}{\partial z^{2}} \right) \bigg|_{b.c} = \frac{\partial^{2} \phi}{\partial z^{2}} \left( \frac{\partial^{2} \phi}{\partial z^{2}} \right) \bigg|_{b.c}$$  \hspace{1cm} \text{(B 18)}$$

Remembering that our initial Laplacian equation for $\Pi$ is equal to zero everywhere except at the point source. Specifically this means that at the boundaries (assuming the point source is not exactly at the boundary):

$$\frac{\partial^{2} \Pi}{\partial \eta^{2}} + \frac{\partial^{2} \Pi}{\partial \xi^{2}} \bigg|_{b.c} = 0 \rightarrow \frac{\partial^{2} \Pi}{\partial \eta^{2}} \bigg|_{b.c} = - \frac{\partial^{2} \Pi}{\partial \xi^{2}} \bigg|_{b.c} = \frac{\partial^{2} \Pi}{\partial \xi^{2}} \bigg|_{b.c}$$  \hspace{1cm} \text{(B 19)}$$

Now comparing $\frac{\partial^{2} \phi}{\partial z^{2}}$, $\frac{\partial^{2} \phi}{\partial z^{2}}$, by using the expressions found for each on the boundaries and the scaling of the problem:

$$\left[ \frac{\partial^{2} \phi}{\partial \eta^{2}} \right] / \left[ \frac{\partial^{2} \phi}{\partial \xi^{2}} \right] = \frac{\left[ \frac{\partial^{2} \phi}{\partial \eta^{2}} \right]}{\left[ \frac{\partial^{2} \phi}{\partial \xi^{2}} \right]} = \left[ \frac{\partial^{2} \phi}{\partial \eta^{2}} \right] / \left[ \frac{\partial^{2} \phi}{\partial \xi^{2}} \right] = \left[ \frac{\partial^{2} \phi}{\partial \eta^{2}} \right] / \left[ \frac{\partial^{2} \phi}{\partial \xi^{2}} \right]$$  \hspace{1cm} \text{(B 20)}$$

Thus also $\frac{\partial^{2} \phi}{\partial z^{2}} < \frac{\partial^{2} \phi}{\partial z^{2}}$ which means that $\frac{\partial^{2} \phi}{\partial z^{2}} \left( \frac{\partial^{2} \phi}{\partial z^{2}} \right) < \frac{\partial^{2} \phi}{\partial z^{2}} \left( \frac{\partial^{2} \phi}{\partial z^{2}} \right)$.
It can be shown that \( \eta \to \infty \) as \( z \to -\infty \) and we can define \( \eta = \eta_0 \) for \( z = 0 \). Thus we now have the second boundary conditions:

\[
\lim_{\eta \to \infty, \eta_0} \frac{\partial \Pi}{\partial \eta} = 0 \quad (B\ 21)
\]

Now we can use Green’s functions theory to solve this equation with Von Neumann boundary conditions on a half plane:

\[
\lim_{\xi \to \pm \infty} \frac{\partial \Pi}{\partial \xi} = 0 \quad (B\ 22a)
\]

\[
\lim_{\eta \to \infty, \eta_0} \frac{\partial \Pi}{\partial \eta} = 0 \quad (B\ 22b)
\]

A similar argument can be made for the SQG case with the point source being located on the boundary.

We can solve a linear PDE with an operator \( \mathcal{L} \) using the following method:

\[
\mathcal{L}\{\Pi\} = F(\eta, \xi) \sim \hat{q}(t) \ \delta(\eta - p, \ \xi - s) \quad (B\ 23)
\]

We are looking for:

\[
\mathcal{L}\{G(\eta, \xi; \ p, s)\} = \delta(\eta - p, \ \xi - s) \quad (B\ 24)
\]

Multiplying by \( F(p, \ s) \) and integrating by \( p, \ s \):

\[
\int \mathcal{L}\{G(\eta, \xi; \ p, s)\}F(p, \ s) \ dp \ ds = \int \delta(\eta - p, \ \xi - s)F(p, \ s) \ dp \ ds = F(\eta, \ \xi) \quad (B\ 25)
\]

Since \( \mathcal{L} \) is a linear operator, acting on \((\eta, \ \xi)\) we can take it out of the integral which acts on \((p, \ s)\):

\[
\mathcal{L}\{\Pi\} = F(\eta, \ \xi) = \int \mathcal{L}\{G(\eta, \xi; \ p, s)\}F(p, \ s) \ dp \ ds = \mathcal{L} \left[ \int G(\eta, \xi; \ p, s)F(p, \ s) \ dp \ ds \right] \quad (B\ 26)
\]

Thus the solution for \( \Pi(\eta, \ \xi) \) is given by:

\[
\Pi(\eta, \ \xi) = \int G(\eta, \xi; \ p, s)F(p, \ s) \ dp \ ds \quad (B\ 27)
\]

Since \( F(\eta, \ \xi) \sim \hat{q}(t) \ \delta(\eta - p, \ \xi - s) \), the Green’s function is proportional to the function \( \to \Pi \sim G \).

We are looking for a solution matching two point sources on a half plane with homogeneous Von Neumann boundary conditions. This means we need to use the method of images in order to find the appropriate solution. In order to satisfy the boundary condition \( \frac{\partial \Pi}{\partial \eta} \bigg|_{\eta = \eta_0} = 0 \), we need to add two image point sources of the same magnitude, sign and distance from the boundary as the point sources located at the front.

The Green’s function matching Von Neumann boundary conditions is:

\[
G(\xi, \eta; \xi', \eta') = -\frac{1}{4\pi} \frac{1}{\sqrt{(\xi - \xi')^2 + (\eta - \eta')^2}} \quad (B\ 28)
\]
In our problem each point source is multiplied by a factor of $D(\xi, \eta, f, t)$, so the total Green’s function which is also the solution for $\Pi$ is:

$$
\Pi(\xi, \eta) = -\frac{1}{4\pi} \frac{q_{ft}}{A_{ft}} \frac{1}{\sqrt{(\xi - \xi_{ft})^2 + (\eta - \eta_{ft})^2}} - \frac{1}{4\pi} \frac{q_{fb}}{A_{fb}} \frac{1}{\sqrt{(\xi - \xi_{fb})^2 + (\eta + \eta_{fb})^2}}
$$

This is the solution for the first order geostrophic potential $\phi^1$ in $\xi, \eta$ space.

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