Can Large Eddy Simulation Techniques Improve Mesoscale Rich Ocean Models?

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The necessary adaptations to large eddy simulation (LES) methods so that they can be used in mesoscale ocean large eddy simulations (MOLES)—where the gridscale lies just below the Rossby deformation radius—are presented. For MOLES, the Smagorinsky model is inappropriate and the similar Leith model is appropriate. The latter is preferable, as the gridscale lies nearer to a potential enstrophy cascade than an energy cascade. A trivial modification of the Leith scheme is shown to improve numerical stability in a global eddy-permitting simulation. Dynamical adjustment of subgrid-scale models—where online diagnosis of eddy terms are used to improve the subgrid-scale model—is introduced and recommended for future investigation.

1. INTRODUCTION

Large-eddy simulations (LES) differ from other fluid flow computations in that the largest eddies are explicitly resolved and the smaller eddies are modeled (as engineers call it) or parameterized (as oceanographers call it). Modeling the ocean and atmosphere inspired the first large-eddy simulations, but true large-eddy simulations of the ocean and atmosphere are surprisingly rare.

Ocean models almost always have eddy parameterizations, be they just larger than natural values of constant diffusivities and viscosities, perhaps oriented along isopycnal surfaces [Redi, 1982], or more sophisticated models such as the eddy extraction of potential energy by along-isopycnal bolus flux model of Gent and McWilliams [1990], perhaps with nonlinear transfer coefficient scaling [Visbeck et al., 1997; Held and Larichev, 1996], the “Neptune” effect parameterizing eddy-topography interaction [Holloway, 1993], or parameterizations based on linear instability analysis [Stone, 1972; Branscome, 1983; Killworth, 2005]. All of these parameterizations are designed for situations where eddies are not present or weak, and, as a result, the model resolution does not appear explicitly. The Herculean—perhaps Sisyphean—task of parameterizing the important effects of eddies in ocean models is done by offline theoretical analysis in lieu of resolved eddies. It is no wonder that there is enormous model sensitivity to the parameterization choices made [e.g., Steiner et al., 2004]. Leaving all to theory seems very unlikely to succeed, as eddies are sensitive to difficult to quantify parameters, such as interaction with unresolved topography [Holloway, 1993], regional variations of hydrodynamic instability [Killworth, 2005], bottom drag [Arbic and Flierl, 2004a; Thompson and Young, 2006], and subtle sensitivity to the Coriolis parameter variation [Arbic and Flierl, 2004b; Thompson and Young, 2007].

Subgrid-scale parameterizations for LES are quite different in character because they attempt only to represent the effects...
of an unresolved cascade of smaller eddies to complement the resolved large eddies. Subgrid-scale parameterizations for LES depend explicitly on the resolution of the model at hand, and their goal is to anticipate higher resolution results at any given resolution. Fortunately, as will be argued quantitatively below, the largest eddies are the ones that do most of the work in stirring tracers both in engineering scale flows and in ocean mesoscale flows. Thus, in LES modeling, results are expected to be less sensitive to subgrid-scale parameterizations than in coarse-resolution modeling. However, even high-resolution ocean models are sensitive to subgrid-scale parameterizations (a frequent topic during sessions like the one on which this volume is based), so parameterization improvement is important. Of course, the range of scales in the ocean is vast and diverse, so phenomena that are substantially smaller than the gridscale—such as submesoscale eddies and microstructure turbulence—will continue to require purely theoretical parameterizations in large-scale models for the next few decades at least [e.g., Fox-Kemper et al., 2008; Fox-Kemper and Ferrari, 2008].

This chapter will first review briefly the traditional subgrid-scale model of engineering-scale 3D LES: the Smagorinsky [1963] nonlinear viscosity. The necessary adaptations appropriate for large-scale ocean models will then be presented. The emphasis will be on ocean models in a larger domain than traditional ocean boundary layer LES [e.g., McWilliams et al., 1997; Wang et al., 1998; Skillingstad et al., 1999], but high-resolution enough to resolve at least the first baroclinic deformation radius. Simulations at this scale will be called mesoscale ocean large-eddy simulations (MOLES). In MOLES, both the tail of the mesoscale eddy spectrum and smaller scale turbulence closures must be provided. The emphasis here is on the former, and it is argued that the mesoscale spectrum can be sensibly represented with a nonlinear diffusivity and viscosity combination adapted from LES methods. However, the Smagorinsky viscosity is not useful for MOLES or in coarser resolution simulations because it relies on the gridscale being in the inertial range of a Kolmogorov [1941] (forward) energy cascade. Smagorinsky’s viscosity might be useful in submesoscale or boundary layer ocean models, but the large-scale high-resolution ocean models that are the primary subject of this book are more likely to have their gridscale in an inertial potential enstrophy cascade of effectively quasi-geostrophic (QG) eddies (if there is an inertial range at all). Models coarser than MOLES tend to not have eddies at all or to have only a weak inverse energy cascade; in these cases, no cascade-based parameterization is useful.

The second section reviews a more ambitious pursuit of subgrid-scale modelers using LES at engineering scales, to use the resolved eddies to improve the subgrid-scale model itself, a process known as dynamical adjustment. Dynamical adjustment has made LES simulations more accurate and reliable with fewer ad hoc assumptions and unknown parameters, beginning with Bardina et al. [1980] and Germano et al. [1991] and reviewed by Meneveau and Katz [2000]. An advantage relevant to MOLES is that dynamical adjustment allows the subgrid-scale parameterization to recognize flow heterogeneity—spatial variation in the natural instabilities of the resolved flow leads to spatial variation in the parameterization as well. For example, dynamical adjustment would allow for a low-viscosity Gulf Stream where it is attached to the coast, and enhanced subgrid-scale effects in the separated boundary current and North Atlantic drift where resolved eddy effects are strong. There is a critical resolution threshold needed for dynamical adjustment: more than one scale of eddies needs to be permitted by the model resolution. Mesoscale eddies at the deformation radius are well represented at O(10 km) resolution, so dynamical adjustment will be possible once O(5 km) resolution is possible. It is argued here that using the adaptations for subgrid-scale parameterization relevant to mesoscale eddies, dynamical adjustment may prove to be a powerful approach in improving subgrid-scale parameterizations for the near future of O(5 km) resolution.

The goal here is to introduce simply the concepts of nonlinear viscosity, nonlinear diffusivity, and dynamical adjustment to an oceanographic audience and to show the adaptations necessary for potential use in future MOLES. Of course, there is much more to LES than just nonlinear viscosities and dynamical adjustment [see, e.g., Sagaut, 2005; Lesieur et al., 2005], and in some circumstances, alternative subgrid-scale parameterizations may be superior [Geurts and Holm, 2003]. Adding a prognostic model for additional unresolved quantities, such as turbulent eddy kinetic energy, has been shown to be useful in LES [Deardorff, 1980; Mellor and Yamada, 1982; Moeng, 1984; Burchard et al., 1998], but implementation of these methods for MOLES is just beginning (A. Adcroft, personal communication). For brevity, the focus here is a presentation we believe will be most palatable to oceanographers and most readily applied to MOLES.

2. NONLINEAR VISCOSITIES AND DIFFUSIVITIES

2.1. The Cascades

The basic principle underlying all of the subgrid-scale parameterizations discussed here is the concept of a cascade through an inertial range. The idea is due to Kolmogorov [1941], who proposes such a scenario for isotropic, homogeneous, 3D turbulence. Subsequent work has shown that similar ideas generalize to isotropic, homogeneous 2D, and QG turbulence as well.
Consider the mean kinetic energy per unit mass (angle brackets are volume mean):

$$
\langle E \rangle = \frac{1}{V} \int \int \int \frac{1}{2} (\mathbf{u} \cdot \mathbf{u}) dV = \int_0^\infty E(k) dk.
$$

$(1)$

$V$ is the domain volume, $\mathbf{u}$ is the velocity, $E(k)$ is the kinetic energy spectrum (assuming homogeneous, isotropic turbulent flow), and $k$ is the wavenumber. The energy spectrum obeys

$$
\frac{\partial E(k)}{\partial t} = - \frac{\partial F_E}{\partial k} - \nu k^2 E(k) + S_E(k, t),
$$

where $F_E$ is the energy flux through a given wavenumber due to nonlinear turbulent interactions, $\nu$ is the molecular viscosity of the fluid, and $S_E(k, t)$ represents sources of kinetic energy from conversion of potential energy by fluid instabilities or from external forces.

Kolmogorov [1941] supposes that in some flows, a quasi-steady state might arise in between the forcing and dissipation, that is, over a range of wavenumber $s$ higher than where $S_E(k)$ is active and lower than where $\nu k^2 E$ is active. If a wavenumber $k$ is assumed to be in this range, then the energy flux through that wavenumber is constant in $k$, as all of the other terms in (2) vanish except $\partial F_E/\partial k$. If the constant value of the flux is denoted by $\varepsilon$, then the assumptions made amount to

$$
\varepsilon = \int_0^{k_*} S_E(k, t) dk = \int_{k_*}^\infty \nu k^2 E(k) dk = F_E(k_*).
$$

$(3)$

Dimensional analysis yields:

$$
[E] = L^3 T^{-2}, \quad [k] = L^{-1}, \quad [\varepsilon] = L^2 T^{-3},
$$

$$
E(k) \propto \varepsilon^{2/3} k^{-5/3}, \quad k_d = \varepsilon^{1/4} \nu^{-3/4}.
$$

$(4)$

(Brackets are used to denote the dimensions of a quantity.) The wavenumber $k_*$ is the energy dissipation wavenumber (sometimes called the Kolmogorov wavenumber), which is large enough for frictional effects to be important.

In 2D flows, there is a second conserved quantity in addition to energy, the enstrophy:

$$
\langle G \rangle = \frac{1}{V} \int \int \int \frac{1}{2} (\nabla_h \times \mathbf{u})^2 dV = \int_0^\infty G(k) dk.
$$

$(5)$

$V_h$ is the horizontal differential operator, $\psi$ is the 2D flow streamfunction ($\mathbf{u} = -V_h \times \psi$), and $G(k)$ is the enstrophy spectrum (assumed isotropic and homogeneous). The enstrophy spectrum has an evolution equation as well,

$$
\frac{\partial G(k)}{\partial t} = - \frac{\partial F_G}{\partial k} - \nu k^2 G(k) + S_G(k, t).
$$

$(6)$

Kraichnan [1967] notes that there might be an inertial enstrophy cascade analogous to the energy cascade where $F_G$ is constant and denoted by $\eta$. Proceeding as above,

$$
\eta = \int_0^{k_*} S_G(k, t) dk = \int_{k_*}^\infty \nu k^2 G(k) dk = F_G(k_*),
$$

$$
[G] = LT^{-2}, \quad [k] = L^{-1}, \quad [\eta] = T^{-3}.
$$

$(7)$

Importantly, as $[\varepsilon]/[\eta] = L^2$, it cannot be the case that both $\varepsilon$ and $\eta$ are constant in $k$ and nonzero. Thus, the presence of an energy cascade precludes the presence of an enstrophy cascade and vice versa.

In sum, there are two limits to consider in 2D turbulence. An inertial energy cascade where

$$
E \propto \eta^{2/3} k^{-5/3}, \quad G \propto \eta^{2/3} k^{1/3}, \quad \eta = 0,
$$

and an inertial enstrophy cascade where

$$
E \propto \eta^{2/3} k^{-3}, \quad G \propto \eta^{2/3} k^{-1}, \quad \varepsilon = 0.
$$

$(8)$

$(9)$

Kraichnan [1967] also considers the wavenumber triads involved in $F_E$ and $F_G$, which suggest that the only way a quasi-steady state can exist is if the energy flux direction is inverse in comparison to 3D turbulence (from small scales to large, i.e., $F_E = -|\varepsilon|$) and the enstrophy flux direction is direct (from large scales to small, $F_G = |\eta|$). Both cascades are envisioned to originate in some central range of forcing wavenumber $s$. In this case, the viscosity is important for dissipating enstrophy—not energy—at small scales and is leading order near the enstrophy dissipation wavenumber

$$
k_d = \eta^{1/6} \nu^{-1/2}.
$$

$(10)$

For Kraichnan’s scenario to work, all of the energy must be dissipated somehow at scales larger than the forcing scales, or there must be no net energy input by forcing (interestingly, with weak dissipation, a flow will often adjust itself to reduce energy and vorticity input [Scott and Straub, 1998; Fox-Kemper, 2004]). Thus, the spectral energy and enstrophy budgets in Kraichnan’s scenario become,
show that in two-vertical-layer QG turbulence, the combination of baroclinic and barotropic flows complicates the dynamics for scales larger than the baroclinic instability scale, but the enstrophy cascade at smaller scales retains the same form as in Kraichnan’s theory. Hua and Haidvogel [1986] show that the turbulence in a six-vertical-mode QG system with non-uniform stratification is not isotropic in the Prandtl-stretched coordinate system. However, one expects that boundary effects will be extremely important with such low vertical resolution, and Charney’s isomorphism between QG and 2D turbulence relies on boundary effects being unimportant.

Another important boundary effect that differentiates QG from 2D turbulence is the presence of temperature modes on the surface that do not require internal QGPV gradients. The study of systems with no interior PV gradients extends back to Eady [1949] and beyond, but study of the dynamics of these surface modes in the ocean has invigorated recently [e.g., Lapeyre and Klein, 2006] along with study of near-surface submesoscale dynamics and frontogenesis. Of importance here is the result of Blumen [1982] that the inertial cascade of surface modes differs from that of due to interior QGPV anomalies shown by Charney [1971] to resemble the 2D turbulent cascade [see also Pierrehumbert et al., 1994; Held et al., 1995]. However, numerical simulations reveal that frontogenesis and mixed-layer eddies may affect the spectral slope near the ocean surface by potential energy conversion at all small scales, which prevents an inertial range of kinetic energy altogether [Capet et al., 2008]. However, these near-surface effects are not well understood presently and have not yet reached the level of a functional parameterization.

Oceanic and atmospheric observations are generally in agreement with the existence of an enstrophy cascade below the deformation radius and away from the surface, but not an inertial inverse energy cascade. Stammer and Wunsch [1995] find an $k^{-5/3}$ slope below the deformation radius and a shallower slope at larger and smaller scales. Their acknowledged uncertainty does not preclude $k^{-5/3} \approx k^{-3}$ in the enstrophy cascade region. In the atmosphere, Nastrom and Gage [1985] find a clear $k^{-3}$ spectral slope below the deformation radius and a shallower slope near $k^{-8/3}$ at smaller scales. The shallow slope at smaller scales might be an inverse cascade up from smaller scales, or an effect of gravity waves, but may result instead by a surface mode cascade [see Tulloch and Smith, 2006, for references]. An inverse energy cascade range does not exist in the atmosphere because the Earth is only slightly larger than the atmospheric deformation radius. There is good evidence of an inverse energy cascade in the ocean, but not an inertial one. The range is limited or nonexistent because the inverse cascade is halted shortly
above the deformation radius for reasons that remain unclear [Scott and Wang, 2005; Thompson and Young, 2007]. The energy flux does decrease rapidly below the deformation radius, which is consistent with an enstrophy cascade, (9), but a small amount of direct energy flux appears to be present, which may be related to interactions between the background internal wave field and the mesoscale QG turbulence (K. Polzin, personal communication). At smaller scale, a shallower slope is observed [Ferrari and Rudnick, 2000] and expected due to surface modes and frontogenesis. LaCasce and Mahadevan [2006] were able to use a shear stress transport-constrained numerical model of only surface modes to obtain a favorable model versus in situ data comparison, suggesting that surface modes are relevant in the near-surface ocean at the submesoscale.

Thus, at some distance away from the boundaries, and below the deformation radius, present understanding predicts the dynamics to be close to QG turbulence in an approximate inertial potential enstrophy cascade. These dynamics are assumed to dominate the gridscale dynamics and form the basis of subgrid-scale parameterizations of MOLES.

2.2. Smagorinsky Viscosity

Smagorinsky [1963] proposed a scaling for an “eddy viscosity” for a numerical model whose gridscale lies in the forward energy cascade range of 3D turbulence as proposed by Kolmogorov [1941]. Consider integrating the energy spectrum equation (2) over all of the wavenumbers that will be resolved in the model, assuming an approximately steady energy content:

$$\frac{d \langle E_\nu \rangle}{dt} = -F_E|_{k_1}^{k*} - \int_0^{k*} \nu k^2 E(k)dk + \int_0^{k*} S_E(k, t)dk.$$  \hspace{1cm} (16)

$$\langle E_\nu \rangle$$ is the resolved kinetic energy per unit mass of the model, and \( k_* = \pi/\Delta x \) is the largest unambiguously resolved wavenumber (the Nyquist wavenumber) of the model grid-scale \( \Delta x \) (assumed to be the same in all directions for now). Asterisks will be used to denote quantities that are to be understood as resolved in the model or resolution-dependent. There is negligible friction at and no energy flux through \( k = 0 \) (largest scales), and the resolved energy and forcing are assumed to be relatively steady, so

$$\int_0^{k*} S_E(k)dk = F_E(k_*) + \int_0^{k*} \nu k^2 E(k)dk.$$  \hspace{1cm} (17)

Resolving the Kolmogorov length scale \( (k_* \ll k_*) \) guarantees that the viscous sink of energy is resolved and then there would be no inertial transfer through the gridscale \( F_E(k_*) = 0 \). However, such fine resolution is impractical in ocean modeling as \( k_* \approx O(1/cm) \) in the ocean.

Smagorinsky proposes to use an eddy viscosity, \( \nu_\nu \), that will result in a resolved Kolmogorov scale and satisfaction of the energy budget (17) with an eddy viscosity term replacing the nonlinear energy flux \( F_E(k_*) \). To determine the value of \( \nu_\nu \), suppose that \( k_* \) lies in the inertial range, so \( F_E(k_*) = \varepsilon \), and recall the scaling for the Kolmogorov wavenumber (4). The Kolmogorov scale based on the increased eddy viscosity is set to be proportional to the gridscale, which yields

$$k_* = \gamma \nu_\nu^{1/4} \nu_\nu^{-3/4},$$  \hspace{1cm} (18)

$$\varepsilon = F_E(k_*) + \int_0^{k*} \nu k^2 E(k)dk \equiv \int_0^{k*} \nu_* k^2 E(k)dk.$$  \hspace{1cm} (19)

The nondimensional coefficient of proportionality between the gridscale and the Kolmogorov length is \( \gamma \), and it is what the user sets.

If the viscous term is evaluated in real space rather than wavenumber space, then

$$\int_0^{k*} \nu_* k^2 E(k)dk = \left< \nu_* S_{\nu_\nu}^{S_\nu S_{\nu_\nu}} \right>,$$  \hspace{1cm} (20)

where \( S_{\nu_\nu} \) are the components of the resolved strain rate tensor \( 2S_{\nu_\nu} = (u_{,ik} + u_{,ik}) \) and angle brackets denote domain average.\(^{12} \) So,

$$\varepsilon = \left< \nu_* S_{\nu_\nu}^{S_{\nu_\nu}} \right>.$$  \hspace{1cm} (21)

Finally, because the turbulence is assumed to be homogeneous, the domain-averaged friction is replaced with a local value, and the result for \( \nu_* \) follows.

$$\nu_* = \left( \frac{\gamma}{k_*} \right)^2 |D_{\nu_*}|,$$  \hspace{1cm} (22)

$$\nu_* = \left( \frac{\gamma \Delta x}{\pi} \right)^2 |D_{\nu_*}|,$$  \hspace{1cm} (23)

$$|D_{\nu_*}| \equiv \sqrt{S_{\nu_\nu}^{S_{\nu_\nu}}}. \hspace{1cm} (24)$$

This assumption of equating domain average to local value is controversial [Lesieur et al., 2005], and it should be kept in mind that the effects of small-scale spatial variation of \( \nu_* \) are not physical. Furthermore, in a model, the viscous term should be in flux form, \( \nabla \cdot \nu_\nu, \nabla \nu \), rather than \( \nu_\nu, \nabla \nu \), so...
momentum will not be spuriously created. For simplicity, the viscosity will be assumed to be smooth enough to approximately commute with derivatives for the remainder of this chapter.

The nondimensional coefficient $\Upsilon$ is generally taken to be in the range 0.5–1 for 3D flows [Smagorinsky, 1993], although numerical instabilities may result in some discretizations for values that small (Griffies and Hallberg [2000] and Griffies [2004] recommend 2.2 to 4).

In a review of the use of the Smagorinsky nonlinear viscosity, Smagorinsky [1993] carefully works through the stress-strain relations in the case of hydrostatic axially symmetric turbulence (in contrast to 3D isotropic turbulence). In this case, there are two viscosities, a horizontal and a vertical. The horizontal viscosity is similar to the 3D isotropic case above,

$$v_{sh} = \frac{\Upsilon_h \Delta x}{\pi} \sqrt{S_{ik} S_{ik}}$$

except the deformation rate appropriate for hydrostatic flows is

$$|D_{s2d}| = \sqrt{\left(\frac{\partial u_x}{\partial x} - \frac{\partial v_y}{\partial y}\right)^2 + \left(\frac{\partial u_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)^2}.$$  

If the horizontal grid is slightly anisotropic, Griffies [2004] suggests the geometric mean of $\Delta x$ and $\Delta y$.

Smagorinsky [1993] shows that a corresponding vertical viscosity should be used:

$$v_{sv} = \frac{\Upsilon_v \Delta z}{\pi} \sqrt{\left(\frac{\partial u_x}{\partial z}\right)^2 + \left(\frac{\partial v_y}{\partial z}\right)^2}$$

The authors are unaware of any use of this vertical viscosity in a climate-scale simulation. Assuming $\Upsilon_z = \Upsilon_v$ (which would be appropriate if the turbulence were 3D isotropic but the gridscale was not), this viscosity would be comparable to typical background values with a 10 km horizontal grid and an 100 m vertical grid.

Smagorinsky’s viscosity is a leap forward in understanding of the interaction of numerical resolution and physics and has proven useful in engineering scale flows since Deardorff [1972], yet it is inappropriate for large-scale ocean and atmosphere simulations as it was initially used [Smagorinsky, 1963]. Smagorinsky [1963] used the earlier work of Kolmogorov [1941], not the later work of Kraichnan [1967] and Charney [1971], who would show that the dynamics relevant at the gridscale of large-scale ocean and atmosphere simulations should be quite different from the scenario envisioned by Kolmogorov. Consider the two possibilities for energy flux in large-scale simulations where OG or 2D turbulence dominates: $\varepsilon = -|\varepsilon|$ in the inverse energy cascade or $\varepsilon = 0$ in the enstrophy cascade. If the Smagorinsky viscosity were to represent the energy flux as in (21), the viscosity would be zero or negative. Furthermore, the Smagorinsky viscosity is proportional to the mean square strain rate $\langle |D| \rangle = \int k^D E(k) dk$ which is not dominated by the smallest scales in an inertial enstrophy cascade $\langle |D| \rangle = \int k^D dk$ as it is in 3D turbulence $\langle |D| \rangle = \int k^D dk$. Thus, the Smagorinsky viscosity will not be adequately sensitive to resolution refinement in large-scale atmosphere-ocean flows. Another approach must be used for MOLES.

2.3. Leith Viscosity

Leith [1996] finds an alternative to the Smagorinsky viscosity by focusing on resolving the direct enstrophy cascade in 2D turbulence rather than the direct energy cascade in 3D turbulence. Integrating the enstrophy spectrum equation (6) over resolved wavenumbers yields:

$$\frac{d\langle G_s \rangle}{dt} = -F_G^{k^*} - \int_0^{k^*} v k^2 G(k) dk + \int_0^{k^*} S_G(k, t) dk,$$

where all notation is the same as above, just with enstrophy swapped for energy. Neglect friction and enstrophy flux through the largest scales, and assuming an approximate steady state yields

$$\int_0^{k^*} S_G(k) dk = F_G(k^*) + \int_0^{k^*} v k^2 G(k) dk.$$  

Again, an eddy viscosity is used to represent the direct inertial cascade with a frictional term whose dissipation lengthscale is resolved.

$$k^* = \Lambda \eta^{1/6} v^{1/2},$$

$$\eta = F_G(k^*) + \int_0^{k^*} v k^2 G(k) dk \equiv \int_0^{k^*} v k^2 G(k) dk.$$  

The nondimensional coefficient $\Lambda$, like $\Upsilon$ above, determines the proportionality of the gridscale to dissipation lengthscale.
Estimating the viscous term in real space by the positive definite friction term in the enstrophy equation yields

$$\int_0^{k^*} \nu_s k^2 G(k) \, dk \simeq \left( \nu_s |\nabla q_{*2d}|^2 \right), \quad (31)$$

This step is a bit more tenuous than in the energy cascade. In the energy cascade, it was argued that the boundary terms vanished with all typical boundary conditions. In the case of vorticity dissipation, the boundary term is proportional to $\nabla^2 q_{*2d}$, which not only does not vanish at the boundary, it is likely to reach its largest values at the boundary where boundary currents act fractionally [Fox-Kemper and Pedlosky, 2004; Fox-Kemper, 2004]. Nonetheless, a positive definite viscosity is required, so the positive definite term must suffice.

Assuming that the gridscale lies in the enstrophy cascade inertial range and making the swap of a local value of friction for a global mean, we find

$$\nu_s = \left( \frac{\Lambda}{k_s} \right)^3 |\nabla q_{2d}| = \left( \frac{\Lambda x^3}{\pi} \right)^3 |\nabla q_{2d}|,$$

$$\nu_s = \left( \frac{\Lambda x^3}{\pi} \right)^3 \nabla_h \left( \frac{\partial u_s}{\partial y} - \frac{\partial v_s}{\partial x} \right). \quad (32)$$

The nondimensional coefficient $\Lambda$ is, like $Y$, an $O(1)$ number.

### 2.4. Biharmonic Forms of Leith and Smagorinsky

Griffies and Hallberg [2000] emphasize that the Smagorinsky scaling produces too much viscosity in MOLES and suggest that a more scale-selective viscosity is needed for QG turbulence. The Leith viscosity (32) is more scale-selective than the Smagorinsky viscosity (23) because it depends on a higher derivative of the velocity. Another way to get enhanced scale selectivity is to use a biharmonic viscosity as first suggested for oceanography by Holland [1978]. A biharmonic viscosity cuts off the energy or enstrophy flux more sharply in wavenumber, but at a cost. Additional artificial boundary conditions must be used with biharmonic viscosities which may strongly affect boundary currents and crucial basin-wide integral balances [Fox-Kemper and Pedlosky, 2004; Fox-Kemper, 2004]. The appropriate boundary conditions on Laplacian eddy viscosity are unclear at best, but at least, the choices of no-stress or no-slip are plausibly physical.

In practice, biharmonic viscosity and diffusivity allow a less viscous, yet numerically stable, simulation than harmonic viscosity and diffusivity. The arguments here for using a viscous turbulence closure to simply truncate the inertial cascade of energy (in the case of Smagorinsky) or enstrophy (in the case of Leith) at a resolved wavenumber seem not to prefer a method of truncation. However, if one considers Laplacian and biharmonic viscosity to be terms in a Taylor series expansion, then both harmonic and biharmonic terms should occur, and the only question is the choice of coefficients. Using biharmonic viscosity alone implies that one zeros the first non-vanishing term in the Taylor series, which is seems unlikely to occur naturally. However, in the context of Smagorinsky and Leith scalings, this objection is less stringent, as one hopes that the details of the truncation of energy and enstrophy are unimportant in comparison with getting the appropriate flow-dependent scaling based on energy and enstrophy flux.

Griffies and Hallberg [2000] propose that one may scale the biharmonic viscosity with the Laplacian viscosity to avoid a computational mode studied by Bryan et al. [1975]. For biharmonic viscosity in a 2D flow, the scaling amounts to

$$\nu_{4*} = \left( \frac{\Lambda x^3}{\pi} \right)^2 \nu_s,$$

where $\nu_{4*}$ is the biharmonic viscosity and $\nu_s$ is the Smagorinsky viscosity (23). The same method generalizes the Leith viscosity (32) to a biharmonic form. However, it should be noted that unlike the harmonic forms, the Griffies and Hallberg [2000] biharmonic scaling does not directly relate to whether energy-dissipation or enstrophy-dissipation scales are resolved. If similar arguments to those above are used to estimate these scales and scale them to the gridscale, the resulting biharmonic viscosities should be:

$$\nu_{4* Smag} = \left( \frac{\Lambda x^3}{\pi} \right)^5 |\nabla^2 u_s|,$$

$$\nu_{4* Leith} = \left( \frac{\Lambda x^3}{\pi} \right)^6 |\nabla^2 q_{*2d}|$$

The scaling differences from Smagorinsky and Leith scaling arise in determining the positive-definite energy- and enstrophy-dissipation operators from $u_s^* \nabla^2 u_s$ and $q_{*2d} \nabla^2 q_{*2d}$. Thus, the biharmonic scaling suggested by Griffies and Hallberg [2000] implies:

$$\sqrt{S_{ik} S_{ik}} \propto \Delta x |\nabla^2 u_s| \quad (34)$$

$$|\nabla q_{*2d}| \propto \Delta x |\nabla^2 q_{*2d}| \quad (35)$$
The assumption thus amounts to the curvature in the velocity and vorticity being dominated by the smallest resolved scales. This assumption can be tested by the expected spectral slopes of the energy and enstrophy cascades. In the case of Smagorinsky viscosity in an energy cascade,

$$\frac{|\nabla^2 u_*|}{\sqrt{S_{xx}S_{xx}}} \propto \sqrt{\int_0^{k_*} k^4 E(k) dk} \propto \sqrt{\int_0^{k_*} k^{7/3} dk} \propto k_*$$

and in the case of Leith viscosity in an enstrophy cascade,

$$\frac{|\nabla^2 q_{2d}|}{|\nabla q_{2d}|} \propto \sqrt{\int_0^{k_*} k^4 G(k) dk} \propto \sqrt{\int_0^{k_*} k^{3} dk} \propto k_*.$$  \hspace{1cm} (36)

Thus, the relevant curvatures are dominated by the grid-scale in both cases. Therefore, up to adjustment of $\gamma$ and $\Lambda$, the relationship between $v_*$ and $v_{a2d}$ suggested by Griffies and Hallberg [2000] is consistent with correct truncation of the enstrophy and energy cascades. Even higher derivative operators will be more and more strongly affected by the grid-scale; thus, the generic enstrophy and energy cascading higher order relation is always

$$v_{2n} = \frac{(\Delta x)^{2n}}{N} v_*.$$  \hspace{1cm} (38)

for any positive integer $n$. The nondimensional tuning factor $N$ is determined by the number of dimensions and the value of $n$, but can always be absorbed into $\Lambda_{a2d}$ and $\gamma_{a2d}$.

The scaling for higher order (38) seems obvious, but it requires confirmation. In fact, Griffies and Hallberg [2000] are in error in proposing the scaling for Smagorinsky in a flow where QG scaling holds and the grid-scale lies in a potential enstrophy cascade range ($E(k) \propto k^{-3}$). In that case, the mean squared strain rate ($\int k^2 E(k) dk$) is dominated by the largest scales, so the integral in the denominator of (36) does not converge. Thus, the assumption required for the biharmonic Smagorinsky viscosity to scale according to numerical stability, (34), fails.

2.5. Modified Leith Viscosity

The Leith viscosity was implemented in the Massachusetts Institute of Technology general circulation model (MITgcm) [Marshall et al., 1997], and initial tests on 2D flows, for example the barotropic wind-driven gyre, were successful. However, during the design of the simulations of nonlinear spindown of a submesoscale front [Boccaletti et al., 2007], where large Rossby numbers and strong internal gravity waves resulted from the initial conditions, one of the authors here noted that when using only Leith viscosity a grid-scale noise pattern emerged that was difficult to remove by increasing $\Lambda$.

Closer analysis revealed that the grid-scale noise pattern was checked in vertical velocity, indicating a divergence/convergence pattern in the horizontal velocity (Figure 1, upper). The Leith viscosity parameterization is derived for purely 2D turbulence where the horizontal flow field is assumed to be divergenceless. However, oceanic flows are only quasi-2D. Divergences in the horizontal velocity are expected to be $O(Ro)$ smaller than the vorticity, a fact that is often used explicitly in scaling arguments to derive “balanced” models where the divergent and vortical horizontal flow are treated separately [e.g., McWilliams, 1985]. However, it is possible that a grid-scale divergence could arise in the model: through numerical errors, through large grid-scale Rossby number ($U/f\Delta x$) and subsequent loss of balance, through forcing, or through internal waves generated by topography or radiated away from large $Ro$ regions. The Leith viscosity only responds to buildup of vorticity at the grid scale, so if this divergent flow happened to have little or no vertical vorticity, it would be totally undamped.

A convenient way to fix this problem is to modify the Leith viscosity to add a damping of the divergent velocity. With introspection, one expects something similar to

$$v_* = \left(\frac{\Delta t}{\pi}\right)^3 \Lambda^6 \left|\nabla_h \partial q_{2d} \right|^2 + \Lambda^5 \left|\nabla_h \cdot (\nabla_h \cdot u)\right|^2.$$  \hspace{1cm} (39)

A physical rationale for this correction is unclear, but the numerical consequences are good. The lower panel of Figure 1 shows that the modified Leith viscosity with $\Lambda_d = \Lambda$ has substantially less checkerboard noise, although the basin mean viscosity is only larger by about 25%. Even doubling $\Lambda$ with $\Lambda_d = 0$ was less effective in reducing the checkerboard pattern, although this doubling increases the viscosity by a factor of eight.

The divergence in MOLES is typically much smaller than the vorticity, so setting $\Lambda_d = \Lambda$ only slightly increases the viscosity. QG scaling indicates [Pedlosky, 1987]

$$\nabla_h \cdot u \approx \frac{\beta v}{f_0} - \frac{\partial q_{2d}}{f_0 \partial t} - \frac{u \cdot \nabla_h q_{2d}}{f_0},$$

$$\nabla_h \nabla_h \cdot u \approx \max\left[O\left(\frac{\beta}{f_0}\right), \frac{1}{f_0}, O\left(\frac{2}{f_0}\right), O\left(\frac{U}{f_0\Delta t}\right)\right].$$
Figure 1. Vertical velocity from a simulation of spindown and instability of a temperature front in a reentrant channel. A simulation with the Leith viscosity applied to the horizontal velocities (upper) and a simulation with the modified Leith viscosity (lower) are shown ($A_y = A$). Light colors are near zero; colors represent upward or downward motion.
Therefore, the added divergence-sensing term will have very little effect on the regions where quasi-geostrophic flow dominates. It will have an impact on high-frequency internal waves, but these are typically not well resolved in MOLES in any case. The near-inertial gravity waves will be affected, but only as strongly as the QG flow. Fronts may have large Rossby number, but the expected increase will only be a factor of \( \sqrt{2} \) in (39), as the divergence and vorticity contributions should match if gradients in only the cross-front direction dominate.

This scaling seems to indicate that one should expect few physical changes due to the added term, yet when this viscosity acts, it acts where the largest values of vertical velocity are. Because the Courant condition on vertical advection \((\Delta t < \Delta z/w)\) is often the numerical constraint that sets the maximum timestep, this viscosity may substantially increase the allowable timestep without severely compromising the simulation. Tests have shown that in some calculations, a timestep three times larger was allowed when \( \Lambda = \Lambda_y = 0 \).

2.6. High-Resolution Global Ocean Simulations

The modified Leith viscosity scheme has also been tested in a high-resolution global ocean MITgcm configuration described by Menemenlis et al. [2005]. This particular configuration employs a cubed-sphere grid projection [Adcroft et al., 2004], which permits relatively even grid-spacing throughout the domain. Each face of the cube comprises 510 \( \times \) 510 grid cells for a mean horizontal grid spacing of 18 km. There are 50 vertical levels ranging in thickness from 10 m near the ocean surface to 450 m near the ocean bottom. Initial temperature and salinity conditions are from the World Ocean Atlas 2001 [Conkright et al., 2002]. Surface boundary conditions are from the National Center for Environmental Prediction (NCEP) and the National Center for Atmospheric Research (NCAR) atmospheric reanalysis [Kistler et al., 2001] and are converted to heat, freshwater, and wind stress fluxes using the Large and Pond [1981, 1982] bulk formulae. Shortwave radiation decays exponentially as per Paulson and Simpson [1977]. Vertical mixing follows the method of Large et al. [1994] with background vertical diffusivity of \( 1.5 \times 10^{-3} \, \text{m}^2 \, \text{s}^{-1} \) and viscosity of \( 10^{-3} \, \text{m}^2 \, \text{s}^{-1} \). A third-order, direct-space-time advection scheme with flux limiter is employed and there is no explicit horizontal diffusivity.

Following a 38-year model spin-up, several additional 1-year (2001) integrations were conducted to test the stability and quality of the modified Leith scheme. Figure 2 displays surface kinetic energy from two such integrations. The first integration uses biharmonic Leith viscosity (LeithOnly, top panel) and the second integration uses biharmonic Leith viscosity modified to sense the divergent flow (LeithPlus, bottom panel). Both test integrations use a time step of 600 s to stabilize the LeithOnly test case and for more direct comparison with the LeithPlus test case. The LeithOnly integration has slightly more volume-averaged kinetic energy, \( 4.36 \times 10^{18} \) vs \( 4.23 \times 10^{18} \) J, but the two simulations are qualitatively very similar.

Figure 3 compares near-surface viscosities from the LeithOnly integration (top panel) to viscosities from the LeithPlus integration (middle panel). Note that the magnitudes and patterns are similar; the global mean differs by less than 20%, and the time-mean vorticity is nearly identical in the two runs. The divergent flow modification (bottom panel) is modest, being typically an order of magnitude smaller than the total viscosity. The impact on model stability, however, and hence on timestep, is significant. It is found that using biharmonic Leith viscosity, the model can be integrated stably using a maximum time step size of 600 s. By comparison, the modified Leith scheme, which includes damping of divergent motions, can be integrated stably using a maximum time step of 1,200 s.

To illustrate this stability issue, Figure 4 shows daily snapshots of maximum Courant number \((w\Delta t/\Delta z)\) in the two simulations. The LeithOnly simulation shows spikes dangerously near 1, which can crash the model, while the LeithPlus simulation has no such spikes. This indicates that in the LeithOnly integration, divergent instabilities occur, which, being unchecked by the Leith viscosity, may render the model unstable. The modified Leith scheme eliminates this problem.

Therefore, in summary, the viscosity for MOLES should be based on an inertial enstrophy cascade that produces the Leith scaling rather than the Smagorinsky scaling resulting from an inertial energy cascade. However, the Leith scaling needs to be adapted so that divergent motions that are present in 3D simulations do not become unstable or overly large to the point that the vertical advection Courant condition is contaminated. Up to this point, the evidence is clear; the remainder of the chapter is more speculative and presents opportunities for improving MOLES in novel ways.

2.7. Nonlinear Diffusivities

Smagorinsky [1963] used equal viscosity and diffusivity, scaled according to (23). He supposed that the stirring by eddies should not distinguish between temperature and velocity at large Reynolds and Péclet number (the Reynolds and Péclet number are the ratio of advection of momentum to friction and advection of temperature to dissipation). This fact, that the eddy Prandtl number (the ratio of eddy viscosity to eddy diffusivity) asymptotes to one at scales much larger
than the frictional and dissipation scales, has been confirmed for 3D turbulence in laboratory and direct numerical simulations (see chapter 15 of Sagaut [2005]). A similar issue arises for diffusivity and viscosity in MOLES, and this section addresses it.

2.7.1. Leith meets Charney. The reasoning behind applying the Leith scaling for viscosity to a 3D flow obeying QG scaling is the similarity of the vorticity equation in the 2D and QG cases (13). This similarity results in similar inverse energy cascades and (potential) enstrophy cascades of the two models. To fully exploit the similarities, however, one should pay attention to the form of the frictional and dissipation terms as well.

In 2D turbulence, the frictional term is easily shown to be

\[ D_{q_{2d}} = \nabla_h \times \mathbf{v} \nabla_h^2 \mathbf{u} = \nu \nabla^2 q_{2d}. \]  

(40)

This result follows from taking the curl of the 2D Navier-Stokes equation. In QG turbulence, the frictional/dissipation operator is not so easily obtained. But, if one forms the relative vorticity equation and then eliminates the vortex stretching term (section 6.5 of Pedlosky [1987]), then one finds

\[ D_{q_{QG}} = \nabla_h \times \mathbf{v}_a \nabla_h^2 \mathbf{u}_a + \frac{\partial}{\partial z} \left( \frac{f^2}{N^2} \nabla \cdot \mathbf{K}_a \cdot \nabla b \right). \]  

(41)
Figure 3. Monthly mean biharmonic viscosity, $v_4$, in the model’s surface level for December 2001. Units are m$^4$ s$^{-1}$ and color scale displays log$_{10}(v_4)$. Top panel is from the LeithOnly integration. Middle panel is from the LeithPlus integration. Bottom panel shows the divergent modification of the LeithPlus integration.
The buoyancy is \( b = g(\rho_0 - \rho) / \rho_0 \). Therefore, the horizontal relative vorticity and vertical buoyancy gradients are connected. If the diffusivity is assumed to be horizontally isotropic with a distinct vertical diffusivity, then

\[
D_{QG} = v_s \nabla^2 q + \frac{\partial}{\partial z} \frac{f^2}{N^2} \nabla \cdot \mathbf{K}_s \cdot \nabla b, \\
b = \frac{\partial q_{QG}}{\partial z}, \\
D_{QG} = v_s \nabla^2 q + \frac{\partial}{\partial z} \frac{f^2 \kappa_h}{N^2} \nabla \frac{\partial q_{QG}}{\partial z} + \frac{\partial}{\partial z} \frac{f^2}{N^2} \kappa_v \frac{\partial^2 b}{\partial z^2}, \\
D_{QG} = \nabla^2 \left[ v_s q + \kappa_h (q_{QG} - q_d) \right] \\
+ \frac{\partial}{\partial z} \frac{f^2}{N^2} \kappa_h \frac{\partial^2 b}{\partial z^2}. \\
\tag{42}
\]

The last step requires the assumption that \( N^2 \) and \( f^2 \) do not change appreciably in the horizontal in comparison to the eddy perturbations (as in QG scaling). In the spirit of truncating the cascade of potential enstrophy cleanly, that is, not breaking \( q_{QG} \) into a part associated with buoyancy diffusion and a part associated with friction, then one would like to assume that the horizontal diffusivity and the horizontal viscosity are equal \( \nu_s = \kappa_s \). The vertical diffusivity remains at this stage, without clear guidance as to how to specify it. It will take a little more work to make clear what to do with it.

QG may be derived in either a \( z \)-level or an isopycnal-layer coordinate framework, and in both cases, the stirring is predominantly horizontal, that is, perpendicular to the vertical coordinate. In QG, the isopycnal slopes must be shallow, and thus, there is little difference between the horizontal velocity and the along-isopycnal velocity. One must be a bit more careful in the primitive equation implementation. Veronis [1977] suggests that using an horizontal diffusivity for eddy parameterizations spuriously mixes adjacent water masses when the isopycnal slope is steep. This mixing effect does not occur in quasi-geostrophic scaling or in primitive equation stirring when buoyancy variance is conserved [Plumb, 1979; McDougall and McIntosh, 2001a, 2001b]. Reducing the Veronis effect greatly improves the water mass properties in ocean models, and avoiding spurious diabatic mixing in eddy parameterizations and advection schemes is part of that process [Roberts and Marshall, 1998; Griffies et al., 2000].

At first, it seems that avoiding spurious diapycnal stirring means that one should interpret the horizontal diffusivity as an along-isopycnal diffusivity and neglect cross-isopycnal diffusivity altogether. In this case, assuming small isopycnal slope [Redi, 1982; Griffies, 1998],

\[
\mathbf{K}_s = \begin{bmatrix}
\kappa_h & 0 & \kappa_h \frac{-b_{z,z}}{b_{x,z}} \\
0 & \kappa_h & \kappa_h \frac{-b_{x,y}}{b_{x,z}} \\
\kappa_h \frac{-b_{z,x}}{b_{x,z}} & \kappa_h \frac{-b_{z,y}}{b_{x,z}} & \kappa_h \frac{b_{x,y}^2 + b_{z,y}^2}{b_{x,z}^2} \\
\end{bmatrix}, \\
\tag{43}
\]

\[
D_{QG} = v_s \nabla^2 q. \\
\tag{44}
\]

The dissipation operator is now independent of buoyancy altogether, as \( \mathbf{K}^* \cdot \nabla b = 0 \). That is, if diffusion is oriented along isopycnals, there is no gradient of buoyancy in the along-isopycnal direction to act on, so there is no diffusive flux.

Roberts and Marshall [1998] suggest that the Gent and McWilliams [1990] parameterization (hereafter GM) may be used as a subgrid-scale mechanism for the dissipation of enstrophy in eddying models. The GM parameterization is not a diffusion of buoyancy along isopycnals, as the vertical
fluxes are reversed in sign to ensure that there is always an extraction of potential energy. However, Griffies [1998] shows that the skew form of GM can be combined with the along-isopycnal diffusivity tensor to give a simple mixing tensor, \( \mathbf{J}_s \), that replaces \( \mathbf{K} \) in the equations above. Assuming that the Redi horizontal stirring and the GM coefficient are equal, and furthermore that they are equal to \( \nu_s \), then

\[
\mathbf{J}_s = \begin{bmatrix}
\nu_{sh} & 0 & 0 \\
0 & \nu_{sh} & 0 \\
\frac{-2b_{s,x}}{b_{s,z}} & \frac{-2b_{s,x}}{b_{s,z}} & \frac{b_{s,x}^2 + b_{s,xy}^2}{b_{s,z}^2}
\end{bmatrix}, \quad (45)
\]

\[
D_{qG} = \nu_s \nabla^2 q_{qG} + \frac{\partial}{\partial z} f^2 \nabla \frac{\partial}{\partial z} \left( \nu_{sh} \frac{b_{s,x}^2 + b_{s,xy}^2}{b_{s,z}^2} \right). \quad (46)
\]

In this case, the horizontal operator acts on the potential vorticity as desired, but now, there is a new term that can be shown to be a sign-definite sink of potential energy (at least in the case of constant \( \nu_s \)). In the GM parameterization, this energy sink is understood to represent the conversion of resolved potential energy to kinetic and potential energy of unresolved baroclinic instabilities.

While this energy extraction effect is desirable numerically and desirable in coarse simulations where no eddies are resolved, it may not be desirable in MOLES. The reason is that an energy flux from resolved to unresolved scales is in conflict with the assumption of no scale to scale energy flux that is required for an inertial enstrophy flux. However, it is a flux of potential, not kinetic, energy, so it does not completely invalidate the kinetic energy arguments above; yet it remains worrisome until a more complete theory of inertial cascades in stratified fluids with the joint effects of the spectra of potential and kinetic energy is developed. It may be that this effect realistically represents the forward energy cascade observed below the dominant instability scale found by Scott and Wang [2005], yet does not represent enough energy flux to upset the idealization of an inertial enstrophy cascade. On the other hand, Capet et al. [2008] find that a similar extraction of potential energy by baroclinic instability allows a powerful forward energy cascade for submesoscale baroclinic instabilities that is able to alter the spectral slope of kinetic energy away from either the Blumen [1982] or the Charney/Kraichnan result discussed above.

In the end, it is unclear to what extent one should worry about parameterizing tracer diffusivities in a model with resolved eddies. Howells [1960] and Young [1987] estimate that the effective diffusivity for stirring of tracers due to isotropic turbulence at smaller scales than \( k_s \) is

\[
\kappa_e(\nu) \approx \sqrt{2 \int_{k_0}^{\infty} k^{-2} E(k)dk}. \quad (47)
\]

In MOLES, large eddies will be resolved \((k_s \ll k)\), and the diffusivity will be dominated by the resolved eddies. This result indicates that one should seek to do as little harm as possible (i.e., spurious mixing) with subgrid-scale diffusive parameterizations. More testing and theory is needed to understand the implications and proper implementation of nonlinear diffusivities in MOLES, but at present, it seems that either using Redi along-isopycnal diffusivity (44) or the GM parameterization (46) is a reasonable handling of tracer diffusivity to complement the Leith scaling of viscosity for MOLES, while using horizontal diffusivity (42) leads to too much spurious mixing of water masses.

3. FILTERING AND DYNAMICAL ADJUSTMENT

The discussion now turns to consideration of dynamical adjustment. It is first necessary to review some of the LES theory on understanding the resolved flow as a filtered form of the total flow.

Often, oceanographers assume that the relationship between the subgrid-scale phenomena and the resolved phenomena is a Reynolds average. A Reynolds average may be considered to be the average over a large ensemble of fluid flows so that the turbulent features are removed. The ensemble average has particular properties: for some variable \( c \), which might be a scalar (e.g., buoyancy) or a vector (e.g., velocity, as used by Reynolds),

\[
c = \bar{c} + c', \quad \bar{c} = \bar{c} \equiv 0. \quad (48)
\]

Reynolds averaging results in advective fluxes that obey:

\[
\mathbf{u} \cdot \nabla c = (\mathbf{u} + \mathbf{u}')(\bar{c} + c') = \mathbf{u} \cdot \bar{c} + \mathbf{R}, \quad (49)
\]

\[
\mathbf{R} \equiv \mathbf{u'} \cdot \mathbf{c'}. \quad (50)
\]

In the atmospheric and oceanographic literature, sometimes a zonal mean is used which has similar properties.

However, since Leonard [1974], the typical approach used for LES differs in that the gridscale is understood to be a spatiotemporal filter instead of a Reynolds average. The filtering approach had been used for LES by Lilly [1967], but Leonard clarified the necessary averaging process. The idea
that the gridscale is a filter on high wavenumbers occurs naturally in the context of spectral models where explicit filters are used to avoid aliasing gridscale noise into the advective fluxes [Orszag, 1971].

Unlike an ensemble mean or a zonal mean, when most filters are applied, they incompletely smooth the gridscale flow. Thus, repeated applications of the filter do not reproduce the same result: \( \bar{\bar{u}} \neq \bar{u} \). Terms that vanish under the Reynolds average do not in the filtered advective flux of \( c \). They are called the Leonard terms, \( \mathbf{L} \), and the cross terms, \( \mathbf{C} \):

\[
\bar{c} = (\bar{u} + u')(\bar{c} + c') = \bar{u}\bar{c} + \mathbf{L} + \mathbf{C} + \mathbf{R},
\]

\[
\mathbf{L} \equiv \bar{u}\bar{c} - \bar{u}\bar{c},
\]

\[
\mathbf{C} \equiv \bar{u}c' + u'\bar{c}'.
\]

The importance of the cross fluxes in a similar context for the ocean is noted by Berloff [2005].

The goal of a subgrid-scale model or eddy parameterization is to approximate the difference between the resolved flux and the filtered total flux: \( \bar{\tau} = \bar{\bar{u}} - \bar{u} \). Under Reynolds averaging, the necessary term is just \( \bar{\bar{u}} = \mathbf{R} \), but under the effects of a filter, the necessary term becomes \( \bar{\tau} = \mathbf{L} + \mathbf{C} + \mathbf{R} \).

Different filters are more or less useful for MOLES. Ensemble filters are expensive, but clearly converge to a Reynolds average. Spectral filters are also Reynolds averaging, but they cannot be used in bounded domains. Time averages are easily implemented and suffer no boundary problems, but mesoscale eddies are not the highest frequency motions in MOLES, so the parameterization will need to represent other high-frequency effects as well, for example internal waves. Spatial filters are easily implemented (e.g., boxcar, Gaussian, 1-2-1), but care is needed when boundaries are within the filter stencil and the cross and Leonard term effects must also be parameterized.

3.1. Dynamical Adjustment I: Bardina Similarity

The arguments in section 2 apply most directly to spatial filtering, and the arguments about replacing the inertial transfer of energy and enstrophy through a given wavenumber were not specific about the form of the inertial transfer, so the whole inertial transfer \( \mathbf{L} + \mathbf{C} + \mathbf{R} \) of momentum could be modeled with the Smagorinsky (23) or Leith (32) viscosities. At first glance, then, it would seem that in practice, there is little difference between assuming a Reynolds average or filtering.

However, Bardina et al. [1980] note that the filtering operation can be used explicitly to allow the subgrid-scale parameterization to depend on the resolved flow. They propose a subgrid-scale model based on a similarity hypothesis between the resolved and unresolved flow:

\[
\bar{u} - \bar{u} \approx \bar{u} - \bar{u}.
\]

This model has distinct advantages over the Smagorinsky and Leith models: it can sense heterogeneity in the flow and can approximate backscatter from the unresolved scales to the large. However, Bardina et al. [1980] found that using just this model was numerically unstable and so added some Smagorinsky viscosity as well to stabilize it. This combination is called the mixed model.

More accurate results can be obtained if a second filter, called the test filter, is applied in addition to the gridscale filter. In this case, the Bardina model becomes

\[
\bar{u} = \bar{u} \approx \bar{u} + \mathbf{G} + \mathbf{F}.
\]

3.2. Dynamical Adjustment II: Germano Identity

Germano [1992] and Germano et al. [1991] take the use of the explicit filter a step farther with the so-called Germano identity:

\[
\mathbf{L} \equiv \mathbf{T} - \bar{\mathbf{T}},
\]

\[
\mathbf{L} \equiv \bar{\mathbf{T}} - \bar{\mathbf{T}},
\]

\[
\mathbf{T} \equiv \mathbf{u}u' - \bar{\mathbf{u}}u' - \bar{\mathbf{u}}u' - \bar{\mathbf{u}}u' .
\]

This identity relates the total subgrid-scale effect at the test filter scale, \( \mathbf{T} \), to the total sub-gridscale effect at the gridscale, \( \bar{\mathbf{T}} \), using only resolved quantities present in a Leonard tensor. Therefore, if the Smagorinsky viscosity is to apply at the test filter level and at the gridscale level, then \( c = u_i \), and

\[
\mathbf{M}_{ij} = \left( \Delta x_i^j \bar{D}^{-i} \bar{S}_{ij} + \Delta x_i^j \bar{D}^{-i} \bar{S}_{ij} \right).
\]
where $\Delta x_\pi$ is the test filter scale and $\Delta x_L$ is the grid scale. This equation is overdetermined for $\gamma$, but Lilly [1992] showed that the minimum-error least-squares solution is

$$\gamma = \frac{\mathcal{K}_{ij} M_{ij}}{\mathcal{M}_{ij} M_{ij}}.$$  \hfill (60)

The Germano identity applies equally well to the Leith viscosity, so

$$\mathcal{L}_{ij} - \frac{1}{3} \mathcal{K}_{kk} \delta_{ij} = \left( \frac{\Lambda}{\pi} \right)^3 \mathcal{M}_{ij},$$

$$\mathcal{M}_{ij} = \left[ \Delta x_L^3 \left| V \right|_3 G S_{ij}^G - \Delta x_L^3 \left| V \right|_3 G S_{ij}^G \right],$$

$$\Lambda = \frac{\pi}{\mathcal{L}_{ij} M_{ij}} \left( \frac{\mathcal{M}_{ij} M_{ij}}{\mathcal{M}_{ij} M_{ij}} \right)^{1/3}. \hfill (61)$$

In the 2D turbulence form that Leith originally proposed, one uses

$$V = \nabla_h q_{2d}.$$ \hfill (62)

The modified Leith form is

$$V = \sqrt{\left( \nabla_h q_{2d} \right)^2 + \left( \nabla_h (\nabla_h \cdot u_\pi) \right)^2}. \hfill (63)$$

Dynamical adjustment requires some smoothing of the coefficients $\gamma$ and $\Lambda$ that are determined by the least-squares method. This smoothing is in line with assuming that the viscosities are sufficiently constant to allow them to be moved into and out of averaging operations (a required step above). This smoothness of viscosity requirement is also suggested above in the derivation of Smagorinsky where the global spectral characteristics of the flow were related to the local strain rate. Optimal methods of smoothing will have to be determined for MOLES.

Dynamical adjustment has been shown to substantially improve the Smagorinsky model when the flow is heterogeneous, for example, where there are boundary currents that transition into turbulent ones, because the parameterization can “turn off” when the resolved flow is laminar. In MOLES, this improvement is expected to be particularly dramatic, as a known bias of existing ocean model using Smagorinsky scaling is overly viscous boundary currents [Jochum et al., 2008].

### 3.3. Dynamical Roberts-Marshall Mixing

Roberts and Marshall [1998] suggest that the GM parameterization and Redi [1982] may be used to cut off the cascade of tracer variance, just as Smagorinsky and Leith are suggested to cut off the cascade of energy and enstrophy. Above, it was argued that a reasonable choice was to set the eddy Prandtl number to 1, equating the GM/Redi coefficients to the Leith viscosity. The Germano identity may be used instead to produce Prandtl numbers other than 1.

If we assume that the GM coefficient and Redi diffusivity have a Leith-like dependence on the gridscale (i.e., quasi-geostrophic scaling) we find,

$$\kappa = \left( \frac{3 \Delta x_L^3}{\pi} \right)^3 \left| \nabla_h q_{2d} \right|,$$

$$\mathcal{L}_i = \left[ \frac{\Gamma}{\pi} \right]^3 \mathcal{M}_i,$$ \hfill (64)

$$\mathcal{M}_i = \left[ \Delta x_L^3 \left| \nabla_h q_{2d} \right|_3 G \nabla_r G - \Delta x_L^3 \left| \nabla_h q_{2d} \right|_3 G \nabla_r G \right],$$

$$\Gamma = \frac{\pi}{\left[ \mathcal{L}_i M_i / \mathcal{M}_i \mathcal{M}_i \right]^{1/3}}. \hfill (65)$$

Coefficients for the GM and Redi fluxes could be treated separately by analyzing the fluxes of buoyancy and salinity and temperature.

The dynamical method also allows estimation of phenomena that are not readily treatable with GM/Redi. For example, it is well-known that in the Eulerian average, cross-isopycnal fluxes result where the buoyancy variance varies in time or space even where the Lagrangian average is adiabatic. This effect is exceedingly difficult to anticipate with a parameterization, but in MOLES, diagnosing the cross-isopycnal fluxes of the resolved eddies can be used to guide the parameterized fluxes. Similarly, Eden et al. [2007] have shown that fluxes that are in the along-isopycnal plane but perpendicular to the horizontal buoyancy gradient have a large effect in a diagnosis of an ocean model. These fluxes are inherently non-local and would be very difficult to parameterize, but in the presence of some resolved eddies, a dynamical method could be attempted.

One very hopeful result from engineering-scale testing of the dynamical adjustment method should be mentioned. Before implementing dynamical adjustment with active feedback on the flow, a priori testing was conducted. That is, the eddy viscosity was diagnosed from high-resolution simulations and compared with the value that would have resulted from the dynamical model scalings. Unexpectedly, using dynamical adjustment with active feedback typically
outperforms the level of accuracy expected from the side-by-side comparison of diagnosed and scaled viscosities in a priori tests [e.g., Vreman et al., 1995].

4. SUMMARY AND DISCUSSION

Large-basin and global ocean models are beginning to be rich with eddies and are beginning to resolve the potential enstrophy cascade below the instability lengthscale. However, they do not resolve all of the scales necessary to completely remove a dependence on subgrid-scale parameterizations. The tools developed for engineering-scale simulations, where the Smagorinsky viscosity is appropriate, may be adapted to the direct enstrophy cascade of mesoscale flows where the Leith viscosity, or a modified version of it, is appropriate. Dynamical adjustment using the Germano identity seems particularly promising, as it may be readily applied to many extant mesoscale parameterizations. Some of these applications are outlined here, and their validation will surely be an interesting future direction in high-resolution ocean models.

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Notes

1. The tensor notation is standard, so indices before the comma represent direction, indices after the comma represent partial derivatives, and Einstein summation is implied (i.e., all repeated indices imply that the term is to be summed over all three directions). See Griffies [2004] for a more detailed presentation.

2. Incidentally, integration by parts converts the expression based on strain rates to the form encountered by dotting the momentum equation with velocity, \( \langle \nu \partial \nabla \nu \rangle \), except for boundary terms that vanish under all typical boundary conditions (periodic, slip, no-slip) or by averaging over a sufficient volume.

3. Salinity and potential temperature are assumed to be diffused similarly and nonlinear equation of state diffusion effects are neglected, both approximations are valid for large Péclet number.

4. In the special case of momentum flux, only the deviatoric stress needs to be modeled because the remaining stresses are solved for along with pressure by enforcing incompressibility.

5. However spurious effects of the z-coordinate advection schemes must be carefully handled [Griffies et al., 2000].

REFERENCES


CAN LARGE EDDY SIMULATION TECHNIQUES IMPROVE MESOSCALE RICH OCEAN MODELS?


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