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Relation Between Structure Functions and Cascade Rates in Anisotropic Two-Dimensional Turbulence

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Relationships between second-order structure functions and the cascade rates of enstrophy and kinetic energy (KE) are presented for the inertial forward enstrophy and inverse KE cascades of two-dimensional (2D) turbulence. These relationships are exact in homogeneous turbulence, and their evaluation does not require any assumption about the isotropy of turbulence. The second-order structure functions are not sign-definite, and their sign detects the direction (upscale or downscale) of the cascades. A corollary relation is derived for the enstrophy cascade, which relates the cascade rate to derivatives of another second-order structure function. The results are also discussed in the context of 2D flows with rotation, mean flow, forcing, and large-scale drag.

Key words:

1. Introduction

One of the few exact laws of turbulence was derived for the inertial kinetic energy (KE) cascade of homogeneous, isotropic, incompressible three-dimensional (3D) turbulence by Kolmogorov (1941). This law is,

$$\overline{\delta u_{\parallel}^3} = -\frac{4}{5}\epsilon r \tag{1.1}$$

where ϵ is the dissipation rate of KE by viscosity, $\delta\phi = [\phi(\mathbf{x}_0+\mathbf{r})-\phi(\mathbf{x}_0)]$ is the difference in a variable ϕ between two locations separated by the vector $\mathbf{r} = r\hat{\mathbf{r}}$, and $u_{\parallel} = \mathbf{u}\cdot\hat{\mathbf{r}}$ is the longitudinal component of the velocity vector $\mathbf{u}$. The overline denotes an averaging operation, which could be over positions $\mathbf{x}_0$, angles, or time. The term on the left-hand side is the third-order longitudinal velocity structure function. Directly measuring ϵ (or equivalently the downscale cascade rate) is challenging because it requires measurements at viscous scales or regularly gridded data that can be spectrally transformed, but Eq. 1.1 provides a method to infer ϵ from the more easily observed structure function.

Kolmogorov (1941) derived Eq. 1.1 under the assumption of isotropy. The same relation was later derived from the Karman-Howarth-Monin equations, governing the spatial autocorrelation of velocity components (Frisch 1995), which allowed the derivation of a more general relationship that applies for anisotropic, but still homogeneous, turbulence,

$$\nabla_r \cdot \left[\overline{\delta\mathbf{u}(\delta\mathbf{u}\cdot\delta\mathbf{u})}\right] = -4\epsilon, \tag{1.2}$$

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where $\nabla_r \cdot$ is the divergence in $\mathbf{r}$-space (Podesta 2008). Equation 1.1 can be derived by assuming isotropy, integrating Eq. 1.2 over a sphere of radius r, and using the divergence theorem. However, under anisotropic conditions Eq. 1.2 cannot be integrated without a detailed knowledge of the anisotropy, which is required to construct an appropriate volume of integration (Galtier 2009*a*). Even if the appropriate volume is known, the resulting laws cannot be represented purely in terms of $u_{\|}$ (Augier *et al.* 2012). Several studies have applied Eq. 1.2 to anisotropic, axisymmetric, 3D turbulence affected by rotation (Galtier 2009*b*), stratification (Augier *et al.* 2012), and magnetism (Galtier 2009*a*, 2011), but all these studies required assumptions be made about the appropriate volume of integration.

Recently, Banerjee & Galtier (2016) found that an alternative exact relationship exists for anisotropic, homogeneous 3D turbulence,

$$\overline{\delta \mathbf{u} \cdot \delta\left(\mathbf{u} \times \boldsymbol{\omega}\right)} = 2\epsilon, \tag{1.3}$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the vorticity. Reassuringly, this same form holds in both non-rotating and constantly rotating coordinate frames, as $\overline{\delta \mathbf{u} \cdot \delta\left(\mathbf{u} \times \mathbf{f}\right)} = \overline{\delta \mathbf{u} \cdot \left(\delta \mathbf{u} \times \mathbf{f}\right)} = 0$, where $\mathbf{f}$ is the vorticity due to the rotating frame (e.g., planetary in the geophysical context; Banerjee & Galtier 2016). Eq. 1.3 provides a relationship between ϵ and a second-order structure function, and it has two benefits over Eqs. 1.1-1.2. First, it applies under anisotropic conditions without requiring integration, and second, estimating second-order structure functions accurately typically requires less data than estimating third-order structure functions (Podesta *et al.* 2009, and examples below).

In contrast to 3D turbulence, two-dimensional (2D) turbulence can have two different inertial cascades; a downscale cascade of enstrophy $[\frac{1}{2}(\boldsymbol{\omega} \cdot \boldsymbol{\omega})]$ at rate ϵ_ω, and an upscale cascade of kinetic energy $[\frac{1}{2}(\mathbf{u} \cdot \mathbf{u})]$ at rate ϵ. In the upscale KE cascade of 2D turbulence Eq. 1.2 applies. For *isotropic and homogeneous* 2D turbulence $\overline{\delta u_{\|}^3}$ is proportional to $-\epsilon r$, as in Eq. 1.2 but with a different constant of proportionality (Lindborg 1999) due to the 2D integration surface being a disk rather than a sphere. In the enstrophy cascade of isotropic homogeneous 2D turbulence, the following laws apply (Lindborg 1996, 1999),

$$\overline{\delta u_{\|}^3} = \frac{1}{8}\epsilon_\omega r^3, \quad \text{and} \quad \overline{\delta u_{\|} \delta\omega \delta\omega} = -2\epsilon_\omega r. \tag{1.4}$$

While the above equations don't apply in the enstrophy cascade of *anisotropic* 2D turbulence, a divergence law can still be formulated (Lindborg 1996),

$$\nabla_r \cdot \left[\overline{\delta \mathbf{u} \delta\omega \delta\omega}\right] = -4\epsilon_\omega. \tag{1.5}$$

The evaluation of Eq. 1.5 in anisotropic flows is challenging for the same reasons as discussed for Eq. 1.2. These 2D relationships were originally developed to study large-scale geophysical systems (e.g. Cho & Lindborg 2001; Lindborg & Cho 2001; Deusebio *et al.* 2014), where often data is sparse and flows are anisotropic (Rhines 1975). Therefore it would be useful to have an analogue of Eq. 1.3 for 2D turbulence, which depends on a second-order, rather than third-order, structure function and is insensitive to anisotropy.

The enstrophy cascade laws of Eqs. 1.4 and 1.5 are analogous to the energy cascade laws of Eqs. 1.1 and 1.2. However, there is currently no 2D turbulence version of Eq. 1.3 for either the enstrophy or inverse KE cascades. The aim of the present paper is to derive the former; a relationship between second-order structure functions and the enstrophy cascade rate, which applies under anisotropic conditions and does not require integration, thus bypassing the need for prior knowledge or assumptions about the anisotropy. In Section 2 we derive this relationship and its corollary, and we discuss the effects of mean flow and system rotation on the relation. The applicability of Eq. 1.3 to the inverse KE cascade of 2D turbulence is covered in Section 3. The utility of the results, including their generalization to 2D systems with forcing and large-scale drag, are discussed in Section 4. The results are summarized in Section 5.

2. Enstrophy cascade of 2D turbulence

Two-dimensional, incompressible flows are governed by the vorticity equation,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = \nu \nabla^2 \boldsymbol{\omega}, \tag{2.1}$$

where $\boldsymbol{\omega}$ is vorticity, $\mathbf{u}$ is velocity, ν is viscosity, and ∇ denotes derivatives with respect to position $\mathbf{x}$. For simplicity we shall initially consider a flow that consists only of turbulence, that is there is no mean flow ($\overline{\mathbf{u}} = 0$), however the results are unaffected by the presence of a constant mean flow ($\overline{\mathbf{u}} = \mathbf{U}_0$), as we shall discuss in Section 2.1.

An equation for the spatial autocorrelation of vorticity $\overline{\boldsymbol{\omega}' \cdot \boldsymbol{\omega}}$, where $\boldsymbol{\omega} = \boldsymbol{\omega}(\mathbf{x})$ and $\boldsymbol{\omega}' = \boldsymbol{\omega}(\mathbf{x} + \mathbf{r}) = \boldsymbol{\omega}(\mathbf{x}')$ can be derived by multiplying Eq. 2.1 by $\boldsymbol{\omega}'$, and the analogous budget for $\boldsymbol{\omega}'$ by $\boldsymbol{\omega}$. Summing the resulting equations together and averaging gives,

$$\frac{\partial \overline{\boldsymbol{\omega} \cdot \boldsymbol{\omega}'}}{\partial t} + \overline{\boldsymbol{\omega}' \cdot (\mathbf{u} \cdot \nabla \boldsymbol{\omega})} + \overline{\boldsymbol{\omega} \cdot (\mathbf{u}' \cdot \nabla' \boldsymbol{\omega}')} = D_\omega(\mathbf{r}), \tag{2.2}$$

where $D_\omega(\mathbf{r}) = \overline{\boldsymbol{\omega}' \cdot (\nu \nabla^2 \boldsymbol{\omega})} + \overline{\boldsymbol{\omega} \cdot (\nu \nabla'^2 \boldsymbol{\omega}')} = 2\nu \nabla_r^2 \overline{(\boldsymbol{\omega} \cdot \boldsymbol{\omega}')}$, ∇' denotes derivatives with respect to position $\mathbf{x}' = \mathbf{x}_0 + \mathbf{r}$, and we have used homogeneity and the fact that $\nabla \to \nabla_{x_0} - \nabla_r$ and $\nabla \to \nabla_r$ under the change of variables $\mathbf{x} = \mathbf{x}_0$ and $\mathbf{x}' = \mathbf{x}_0 + \mathbf{r}$ (see Appendix A or Lindborg & Cho 2001; Lindborg 2015).

It is then convenient to define a variable which is equivalent, but opposite, in magnitude to the advection of vorticity, $\boldsymbol{\mathcal{A}}_\omega = -\mathbf{u} \cdot \nabla \boldsymbol{\omega}$. In 2D dynamics, $\mathcal{A}_\omega$ can also be written in several other forms. For example using vector identities it can be shown that,

$$\boldsymbol{\mathcal{A}}_\omega = \mathbf{u} \times (\nabla \times \boldsymbol{\omega}) = -\mathbf{u} \times \nabla^2 \mathbf{u} = -\mathbf{u} \cdot \nabla \boldsymbol{\omega} = J(\omega, \psi), \tag{2.3}$$

where J denotes the 2D Jacobian, and ψ is the streamfunction of the flow. The utility of each of these formulations could vary for differing experimental setups, datasets, or numerical convenience, and the first formulation is similar to the Lamb vector ($\mathbf{u} \times \boldsymbol{\omega}$) in Eq. 1.3. With this variable, Eq. 2.2 and its single-point limit can respectively be written,

$$\frac{\partial \overline{\boldsymbol{\omega} \cdot \boldsymbol{\omega}'}}{\partial t} = \overline{\boldsymbol{\omega}' \cdot \boldsymbol{\mathcal{A}}_\omega} + \overline{\boldsymbol{\omega} \cdot \boldsymbol{\mathcal{A}}'_\omega} + D_\omega(\mathbf{r}), \quad \text{and} \tag{2.4}$$

$$\frac{\partial \overline{\boldsymbol{\omega} \cdot \boldsymbol{\omega}}}{\partial t} = \frac{\partial \overline{\boldsymbol{\omega}' \cdot \boldsymbol{\omega}'}}{\partial t} = 2\overline{\boldsymbol{\omega} \cdot \boldsymbol{\mathcal{A}}_\omega} - 2\epsilon_\omega, \tag{2.5}$$

where $\epsilon_\omega = \nu \overline{(\partial \omega_i / \partial x_j)^2}$ is the enstrophy cascade rate and we have used homogeneity ($\overline{\boldsymbol{\omega} \cdot \boldsymbol{\omega}} = \overline{\boldsymbol{\omega}' \cdot \boldsymbol{\omega}'}$ and $\overline{\boldsymbol{\omega} \cdot \boldsymbol{\mathcal{A}}_\omega} = \overline{\boldsymbol{\omega}' \cdot \boldsymbol{\mathcal{A}}'_\omega}$). Subtracting Eq. 2.4 from Eq. 2.5 and re-arranging we find,

$$\frac{\partial \overline{\delta \boldsymbol{\omega} \cdot \delta \boldsymbol{\omega}}}{\partial t} = 2\overline{\delta \boldsymbol{\omega} \cdot \delta \boldsymbol{\mathcal{A}}_\omega} - 4\epsilon_\omega - 2D_\omega(\mathbf{r}). \tag{2.6}$$

Assuming that there are a range of inertial cascade scales where turbulent statistics are stationary and $D_\omega(\mathbf{r}) \approx 0$ (Lindborg 1996), we arrive at the following equation for the inertial enstrophy cascade of 2D turbulence,

$$\overline{\delta \boldsymbol{\omega} \cdot \delta \boldsymbol{\mathcal{A}}_\omega} = 2\epsilon_\omega. \tag{2.7}$$

This is the main result of this paper. This equation applies under anisotropic conditions, and does not require integration over a specific surface. It is analogous to Eq. 1.3 for 3D turbulence (Banerjee & Galtier 2016). If the co-ordinate system is chosen such that vorticity is aligned with the vertical vector ($\hat{\mathbf{z}}$), then $\boldsymbol{\omega} = (0, 0, \omega)$, $\mathbf{u} = (u, v, 0)$, and the LHS of Eq. 2.7 can alternatively be written $-\overline{\delta\omega\delta(u\partial_x\omega + v\partial_y\omega)} = \overline{\delta\omega\delta\mathcal{A}_\omega}$.

Comparing Eqs. 2.7 and 1.5 it is apparent that the second- and third-order structure functions must be related by $\overline{\delta\omega\delta\mathcal{A}_\omega} = -(1/2)\nabla_r \cdot (\overline{\delta\mathbf{u}\omega\omega})$. This relationship can be validated by noting that,

$$\begin{aligned}\nabla_r \cdot (\overline{\delta\mathbf{u}\delta\omega\delta\omega}) &= \nabla_r \cdot (\overline{\mathbf{u}'\omega'\omega'} - 2\overline{\mathbf{u}'\omega\omega'} - \overline{\mathbf{u}\omega'\omega'} + \overline{\mathbf{u}'\omega\omega} + 2\overline{\mathbf{u}\omega'\omega} - \overline{\mathbf{u}\omega\omega}) \\ &= -2\nabla' \cdot (\overline{\mathbf{u}'\omega\omega'}) + \nabla \cdot (\overline{\mathbf{u}\omega'\omega'}) + \nabla' \cdot (\overline{\mathbf{u}'\omega\omega}) - 2\nabla \cdot (\overline{\mathbf{u}\omega'\omega}) \\ &= -2\overline{\omega\mathbf{u}' \cdot \nabla'(\omega')} - 2\overline{\omega'\mathbf{u} \cdot \nabla(\omega)} \\ &= 2\overline{\delta\omega\delta(u\partial_x\omega + v\partial_y\omega)} = -2\overline{\delta\omega\delta\mathcal{A}_\omega},\end{aligned} \tag{2.8}$$

where we have used co-ordinate transforms (Appendix A), incompressibility and homogeneity (i.e., $2\nabla \cdot (\overline{\mathbf{u}\omega\omega}) = \overline{\omega\mathbf{u}\nabla \cdot \omega} = \overline{\omega'\mathbf{u}'\nabla' \cdot \omega'} = 0$), but isotropy is not assumed or required.

2.1. *Effects of Rotation and Mean Flow*

If a 2D turbulent system is rotating at a constant rate $\boldsymbol{\Omega}$, the fluid parcels are affected by a Coriolis acceleration. As a result, in a rotating system the vorticity budget (Eq. 2.1) becomes a budget for the absolute vorticity oriented out of the plane of motion ($\zeta = \omega + 2\boldsymbol{\Omega} \cdot \hat{\boldsymbol{z}}$) which includes a Coriolis term $\nabla \times [\mathbf{u} \times 2\boldsymbol{\Omega}]$. However, noting that such systems obey governing equations identical to (2.1) except with $\omega \to \zeta$, the immediate equivalent to (2.7) is

$$\overline{\delta\boldsymbol{\zeta} \cdot \delta\boldsymbol{\mathcal{A}}_\zeta} = 2\epsilon_\zeta, \qquad \boldsymbol{\mathcal{A}}_\zeta = -\mathbf{u} \cdot \nabla\zeta = \mathbf{u} \times (\nabla \times \boldsymbol{\zeta}). \tag{2.9}$$

The effects of system rotation on this relation can then be diagnosed; for example, for constant frame rotation (f-plane, $\boldsymbol{\zeta} = (\omega + f_0)\hat{\boldsymbol{z}}$) or tangent plane [$\beta$-plane, $\boldsymbol{\zeta} = (\omega + f_0 + \beta y)\hat{\mathbf{z}}$] systems. For 2D motions on the surface of a rotating sphere [$\boldsymbol{\zeta} = (\omega + 2\Omega\sin\theta)\hat{\mathbf{z}}$, θ is latitude] the scales of interest would need to be small enough that $\delta\boldsymbol{\zeta}$ and $\delta\mathcal{A}_\zeta$ are dominated by turbulent fluctuations rather than spatial variation of the local vertical direction ($\hat{\mathbf{z}}$).

It was stated at the start of the above derivation that Eq. 2.7 is valid in turbulent flows with a constant background velocity. The presence of a constant background mean flow $\mathbf{U}_0$, where $\mathbf{u} = \mathbf{u}_t + \mathbf{U}_0$ and $\overline{\mathbf{u}_t} = 0$ is the turbulent velocity field, would lead to a term, $\mathbf{U}_0 \cdot \nabla\boldsymbol{\omega}$, on the left-hand side of Eq. 2.1 (note that $\overline{\boldsymbol{\omega}}$ is still zero). As a result, Eq. 2.7 would have an additional term that, for a constant background flow, is zero;

$$\begin{aligned}\overline{\delta\boldsymbol{\omega} \cdot \delta\left(\mathbf{U}_0 \cdot \nabla\boldsymbol{\omega}\right)} &= \mathbf{U}_0 \cdot \nabla\left(\frac{1}{2}\overline{\boldsymbol{\omega} \cdot \boldsymbol{\omega}} - \overline{\boldsymbol{\omega}' \cdot \boldsymbol{\omega}}\right) + \mathbf{U}_0 \cdot \nabla'\left(\frac{1}{2}\overline{\boldsymbol{\omega}' \cdot \boldsymbol{\omega}'} - \overline{\boldsymbol{\omega}' \cdot \boldsymbol{\omega}}\right), \\ &= -\mathbf{U}_0 \cdot \nabla\left(\overline{\boldsymbol{\omega}' \cdot \boldsymbol{\omega}}\right) - \mathbf{U}_0 \cdot \nabla'\left(\overline{\boldsymbol{\omega}' \cdot \boldsymbol{\omega}}\right), \\ &= \mathbf{U}_0 \cdot \nabla_r\left(\overline{\boldsymbol{\omega}' \cdot \boldsymbol{\omega}}\right) - \mathbf{U}_0 \cdot \nabla_{x_0}\left(\overline{\boldsymbol{\omega}' \cdot \boldsymbol{\omega}}\right) - \mathbf{U}_0 \cdot \nabla_r\left(\overline{\boldsymbol{\omega}' \cdot \boldsymbol{\omega}}\right), \\ &= -\mathbf{U}_0 \cdot \nabla_{x_0}\left(\overline{\boldsymbol{\omega}' \cdot \boldsymbol{\omega}}\right) = 0,\end{aligned} \tag{2.10}$$

where we have used homogeneity and a coordinate transformation. Equation 2.7 is therefore not affected by a constant background flow. Equivalently it is Galilean invariant - unaffected by the relative velocities of the fluid and observational platforms. This is an important property for the interpretation of remotely-sensed data (e.g., satellite observations of the Earth).

2.2. *A corollary in the enstrophy cascade*

In isotropic, homogeneous 2D turbulence, there are two relationships for the third-order structure functions in the enstrophy cascade (Eq. 1.4). In analogy to this, another second-order structure function relationship can be derived in addition to Eq. 2.7. The second-order vorticity structure function budget (Eq. 2.6) in homogeneous 2D turbulence can be written in several different forms, and comparing these different forms it is seen that $\overline{\delta\boldsymbol{\omega}\cdot\delta\boldsymbol{\mathcal{A}}_\omega} = -(1/2)\nabla_r\cdot\left[\overline{\delta\mathbf{u}(\delta\boldsymbol{\omega}\cdot\delta\boldsymbol{\omega})}\right]$ (Lindborg 1996, their Eq. 4.3), and $\overline{\delta\boldsymbol{\omega}\cdot\delta\boldsymbol{\mathcal{A}}_\omega} = -(1/2)\nabla_r^2\left[\nabla_r\cdot\left(\overline{\delta\mathbf{u}\,(\delta\mathbf{u}\cdot\delta\mathbf{u})}\right)\right]$ (Lindborg 1999, their Eq. 12). For homogeneous (3D or 2D) turbulence it has also been demonstrated that $2\overline{\delta\mathbf{u}\cdot\delta\,(\mathbf{u}\times\boldsymbol{\omega})} = -\nabla_r\cdot\left(\overline{\delta\mathbf{u}\,(\delta\mathbf{u}\cdot\delta\mathbf{u})}\right)$ (Banerjee & Galtier 2016). Equation 2.7 can then be written in an alternative form,

$$\nabla_r^2\left[\overline{\delta\mathbf{u}\cdot\delta\,(\mathbf{u}\times\boldsymbol{\omega})}\right] = 2\epsilon_\omega. \tag{2.11}$$

Together, Eq. 2.7 and Eq. 2.11 provide relationships between second-order structure functions and the enstrophy cascade rate in the enstrophy cascade of 2D turbulence. These equations apply under anisotropic conditions, with the important benefit that Eq. 2.7 does not require integration, unlike existing relations (Eq. 1.5). While Eq. 2.11 does require integration, in the limit of isotropic turbulence it can be integrated over a disc to find,

$$\overline{\delta\mathbf{u}\cdot\delta\,(\mathbf{u}\times\boldsymbol{\omega})} = \frac{1}{2}\epsilon_\omega r^2. \tag{2.12}$$

Equations 2.11 and 2.12 can also be modified to account for the effects of a rotating system by changing $\boldsymbol{\omega}\to(\boldsymbol{\omega}+2\boldsymbol{\Omega})$. As in Eq. 2.7, this rotation effect is identically zero under solid body rotation.

3. Inverse Kinetic Energy Cascade

In addition to an enstrophy cascade, 2D turbulence can also produce an inverse cascade of kinetic energy (KE), where 'inverse' refers to the fact that KE moves from small scales to large scales. Except for the constraint that 2D flow only has two velocity components, the 2D and 3D Navier-Stokes equations are identical,

$$\frac{\partial\mathbf{u}}{\partial t} = -\mathbf{u}\cdot\nabla\mathbf{u} + \nu\nabla^2\mathbf{u} - \nabla\left(\frac{p}{\rho}\right) + \mathbf{f}_u = \mathbf{u}\times\boldsymbol{\omega} + \nu\nabla^2\mathbf{u} - \nabla\left(\frac{p}{\rho} + \frac{\mathbf{u}\cdot\mathbf{u}}{2}\right) + \mathbf{f}_u, \tag{3.1}$$

where p is pressure and ρ is (constant) density.

The derivation of Eq. 1.3 for the forward KE cascade of 3D turbulence by Banerjee & Galtier (2016) was derived from Eq. 3.1 by assuming statistical homogeneity and incompressibility, and *without* assuming isotropy. As a result, their derivation can be applied to 2D dynamics, that is,

$$\overline{\delta\mathbf{u}\cdot\delta\,(\mathbf{u}\times\boldsymbol{\omega})} = 2\epsilon = -2P, \tag{3.2}$$

is also a constraint in homogeneous 2D turbulence, where $P = -\varepsilon$ is the upscale cascade rate of KE. Care should be taken to note that in the KE cascade of 2D turbulence, KE is transferred to large scales which means that ϵ and, as a result, $\overline{\delta\mathbf{u}\cdot\delta\,(\mathbf{u}\times\boldsymbol{\omega})}$ are both negative (for this reason $P = -\epsilon$ is often used; Lindborg 1999). This is in contrast to the KE cascade of 3D turbulence where both are positive (Banerjee & Galtier 2016) and KE is transferred to small scales where it is dissipated by viscosity. As discussed by Banerjee & Galtier (2016), in a rotating system $\boldsymbol{\omega}\to\boldsymbol{\omega}+\boldsymbol{\Omega}$ in Eq. 3.2, which has no effect under solid body rotation. They also found that if there is a mean velocity $(\mathbf{U}_0)$ then there is and additional term on the left of Eq. 1.3 which takes the form $\overline{\delta\mathbf{u}\cdot\delta\left(\mathbf{U}_0\times\boldsymbol{\omega}\right)}$. In 2D turbulence this term must be zero as the velocity and vorticity vectors are perpendicular. Therefore Eq. 3.2 (and similarly Eq. 2.11) is applicable even if there is a constant mean velocity.

4. Utility of New Relationships

The above results were derived from the unforced vorticity budget (Eq. 2.1) with viscous damping. It is common for 2D turbulent systems to also be forced, often at a particular scale, or damped at large scales. Forcing can supply energy (enstrophy) for the inverse energy (direct enstrophy) cascade. Previous studies have found that the presence of external forcing at a specific scale does not affect the third-order structure function laws in the forward enstrophy and inverse KE inertial cascades of 2D turbulence (Lindborg 1999; Cerbus & Chakraborty 2017), or the forward energy inertial cascade of 3D turbulence (Banerjee & Galtier 2016). Following the above discussion of the links between third- and second-order structure functions (Sec. 2.2) it follows that the presence of external forcing at a given scale should not affect the second-order structure function laws presented in this paper, providing the scales $(\mathbf{r})$ are within an inertial cascade. Similarly, large-scale drag should not affect the equations for the enstrophy cascade presented here, although it will reduce the down-scale enstrophy cascade rate ϵ_ω as some enstrophy will be removed by drag at large-scales (Cerbus & Chakraborty 2017). In contrast, these laws would need modification in systems with overlapping energy and enstrophy inertial cascades, which can arise from simultaneous small-scale KE forcing and large-scale enstrophy forcing (Lindborg 1999) or from forced 2D turbulence with large-scale or scale-independent drag where enstrophy transfer to large-scales overlaps with the inverse KE cascade (Cerbus & Chakraborty 2017; Thompson & Young 2006). Third-order structure function laws have also been proposed for regions outside the inertial cascades in isotropic 2D turbulence (Xie & Bühler 2018). The relationship shown in Eq. 2.8 means that any extant laws containing $\nabla_r\cdot\left(\overline{\delta\mathbf{u}\delta\omega\delta\omega}\right)$ can be converted into laws in terms of $\overline{\delta\omega\delta\mathcal{A}_\omega}$, and similarly the relation $\overline{\delta\mathbf{u}\cdot\delta\left(\mathbf{u}\times\boldsymbol{\omega}\right)} = -(1/2)\nabla_r\cdot\left(\overline{\delta\mathbf{u}[\delta\mathbf{u}\cdot\delta\mathbf{u}]}\right)$ (Banerjee & Galtier 2016) allows the conversion of extant laws with third-order velocity structure functions into laws with second-order structure functions. Together these relations can be used to convert the above-referenced results into second-order structure function laws.

The relationships derived above depend on second-order structure functions, and may therefore be expected to converge faster than their third-order counterparts (Eq. 1.1, 1.2, 1.4, 1.5 Banerjee & Galtier 2016). However, this intuition follows from the fact that $\overline{\delta\phi\delta\phi}$ and $\overline{\delta\phi\delta\phi\delta\phi}$ are averages of positive-definite and dual-signed quantities respectively. The latter is therefore a residual and requires more data to converge than the former, particularly for variables that are almost symmetrically distributed such as turbulent velocity (Podesta *et al.* 2009). The second-order structure functions presented here are not positive-definite, for example $\delta\boldsymbol{\omega}$ and $\delta\boldsymbol{\mathcal{A}}_\omega$ are not necessarily aligned, and the implications of this for convergence are not trivial. As a preliminary test of this convergence, we diagnosed the different estimators of the enstrophy flux (Eqs. 1.4, 1.5, and 2.7) from a simple isotropic, spectral decaying 2D turbulence simulation and synthetic incompressible flow fields of varying spectral slope. While all three forms are equivalent analytically, it was typically the case that Eq. 2.7 required less sampling than Eqs. 1.4 and 1.5 to converge in this idealized simulation. Differences in convergence result in part from the dual-signed nature of the new structure functions, and partly from the differing mixtures of intervals $(\delta\phi)$ and derivatives with respect to the relative (∇_r) or the absolute (i.e. ω, ∇) co-ordinates. Eq. 1.4 involves 2 ω's and three intervals; Eq. 1.5 involves an overall derivative, 2 ω's and three intervals; and Eq. 2.7 involves two ω's, one of which is differentiated again, and two intervals. As noted by Feraco *et al.* (2018), an interval is just a definite integral of a derivative. Future studies should quantify the convergence of these statistics in more detail, particularly in anisotropic flows where Eq. 2.7 is seems to be accurate but less precise, while Eqs. 1.4 and 1.5 are not accurate but may be less noisy.

Third-order structure functions have previously been used to study the dynamics of the atmosphere and ocean (Cho & Lindborg 2001; Deusebio *et al.* 2014; Balwada *et al.* 2016; Poje *et al.* 2017). However, making appropriate measurements of flow properties can be challenging due to the nature of both in situ and remote data collection away from the land surface (LaCasce 2008; Pearson *et al.* Under Review). Although second-order structure functions may require less data to converge with their mean value than third-order structure functions, the results presented here require products of flow properties and their derivatives (such as vorticity). These properties may be difficult to infer from some geophysical measurement techniques at quasi-2D scales. Despite this, the results could still be useful for observations which include sufficient information and for the analysis of numerical simulations, which can presently resolve large-scale turbulence in both the atmosphere and ocean at global scales (Pearson *et al.* 2017). Analyses of cascade rate statistics in these simulations can improve our physical understanding of these systems (e.g., Pearson & Fox-Kemper 2018).

5. Conclusions

Two new structure function laws have been derived for the enstrophy cascade of anisotropic, homogeneous 2D turbulence (Eq. 2.7 and Eq. 2.11). One of these laws relates a second-order structure function to the enstrophy cascade rate, and in contrast to existing laws it can be evaluated *without* integration. This means that it does not require a prior quantification of flow anisotropy. The results are not affected by a constant background flow or by solid body rotation of the system. The second-order mixed structure functions used here can be related to third-order structure functions. As a result, relationships follow for more complex 2D turbulent systems where third-order structure function laws already exist, such as those with large-scale drag or forcing at a specific scale.

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Appendix A. Coordinate Transformations

The derivation of the correlation equations and structure function relationships requires a coordinate transformation to relative coordinates, $\mathbf{r}$ and $\mathbf{x}_0$, from the Navier-Stokes equation co-ordinates, $\mathbf{x} = \mathbf{x}_0$ and $\mathbf{x}' = \mathbf{x}_0 + \mathbf{r}$. This transformation can be derived from the total derivative of a function $f = f(\mathbf{x}, \mathbf{x}') = f(\mathbf{x}_0, \mathbf{r})$ (for simplicity we assume differences in an arbitrary direction and drop vector notation);

$$df = \left(\frac{\partial f}{\partial x}\right)_{x'} dx + \left(\frac{\partial f}{\partial x'}\right)_{x} dx' = \left(\frac{\partial f}{\partial x_0}\right)_{r} dx_0 + \left(\frac{\partial f}{\partial r}\right)_{x_0} dr, \tag{A 1}$$

where subscripts on brackets denote variables that are held constant and df denotes the total differential of f. It then follows that, if x is varied while x' is held constant

($dx' = 0 = dx_0 + dr$ and $dx = dx_0$) then,

$$\left(\frac{\partial f}{\partial x}\right)_{x'} = \left(\frac{\partial f}{\partial x_0}\right)_r - \left(\frac{\partial f}{\partial r}\right)_{x_0} . \tag{A 2}$$

Similarly, if instead x' is varied while x is held constant ($dx = 0 = dx_0$ and $dx' = dx_0 + dr = dr$) then,

$$\left(\frac{\partial f}{\partial x'}\right)_{x} = \left(\frac{\partial f}{\partial r}\right)_{x_0} . \tag{A 3}$$

The above two equations demonstrate the co-ordinate transformation of Lindborg & Cho (2001); $\partial/\partial x \to \partial/\partial x_0 - \partial/\partial r$ and $\partial/\partial x' \to \partial/\partial r$. The same method can be applied to demonstrate other transformation choices (Lindborg 2015). For example the transform from r and x_0 to x and x' is $\partial/\partial r \to \partial/\partial x'$ and $\partial/\partial x_0 \to \partial/\partial x + \partial/\partial x'$. In this transformation, if the flow is homogeneous $\partial\overline{f}/\partial x_0 = 0$ for any averaged variable $\overline{f}$, resulting in the additional relations $\partial\overline{f}/\partial x = -\partial\overline{f}/\partial x'$ and $\partial\overline{f}/\partial r = -\partial\overline{f}/\partial x$. Note that f in this case can be a function of both primed and unprimed variables.

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