Nonequilibrium oscillations, probability angular momentum, and the climate system

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Abstract. The climate system is, apart from anthropogenic forcing and other external perturbations, in a thermodynamically nonequilibrium steady-state. A major difference between fluctuations in nonequilibrium and equilibrium steady-states is that noneqilibrium steady-states violate time reversal symmetry. One manifestation is the presence of persistent probability currents, which form closed loops. In this paper, we regard the natural variability of the climate system as an expression of nonequilibrium fluctuations within the statistically steady-state and study the consequences of such persistent current loops. In particular, they give rise to preferred spatio-temporal patterns of natural climate variability that take the form of climate oscillations such as the El-Niño Southern Oscillation (ENSO) and the Madden-Julien Oscillation (MJO). In the phase space of climate indices, we observe persistent currents and define a new diagnostic for these currents: the probability angular momentum ($\mathcal{L}_\tau$). Using the climate time series of ENSO and MJO, we compute both the averages and the distributions of $\mathcal{L}_\tau$. These results are in good agreement with the analysis from a linear Gaussian model. We propose that probability angular momentum provides a new quantification of climate oscillations across models and observations.

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1. Introduction

Nearly all the interesting phenomena around us emerge from tractable interactions between simple constituents, e.g., electromagnetism and atoms. However, to understand how emergent phenomena arise - the goal of statistical mechanics - is extremely challenging, mainly due to the presence of the large number of degrees of freedom and the large consequences of small deviations from symmetry. For systems in thermal equilibrium, Boltzmann and Gibbs provided a highly successful framework, while linear response theory is adequate for describing systems *near* equilibrium, see e.g., [1]. Yet, most fascinating phenomena in nature are associated with systems driven far from equilibrium, e.g., all life forms, sociopolitical structures, and the global climate. In particular, such system cannot exist under conditions for thermal equilibrium, i.e., when they are totally isolated or allowed to exchange energy (or particles, or information [2]) with just one reservoir. Despite much progress on fluctuation theorems and the "nonequilibrium counterpart" of the free energy in recent years (see, e.g., Ref. [3]), an overarching framework for far-from-equilibrium systems remains elusive. Often, to study such interesting systems, we rely on models with a few (macroscopic) degrees of freedom, evolving as nonequilibrium stochastic processes. One frequently used approach involves master equations for the probability distribution, with time-independent rates. While analyzing the full time dependence is generally beyond our reach, we can take initial steps, by studying the associated stationary states (which are guaranteed to exist). If these rates obey detailed balance, the stationary state can be easily found and we can treat them as systems in thermal equilibrium ([1]). On the other hand if the rates *violate* detailed balance, then even finding the stationary distribution is highly non-trivial in general [4]. Specifically, such detailed balance-violating systems may settle into nonequilibrium steady-states (NESS), and understanding their properties (e.g., fluctuations and correlations) is quite challenging. In particular, unlike systems in thermal equilibrium, there are persistent probability currents [5], which form closed loops and characterize underlying rotations in phase space. Studying the observable consequences of such steady current loops is surely a valuable endeavor, likely leading to fruitful insights for all NESS. Here we focus one such observable - the probability angular momentum, in analogy with the familiar angular momenta associated with fluid current loops (e.g., [6]). As shown below, this quantity is intimately related to fluctuations and temporal correlations in the NESS. Introduced recently in other contexts [7, 8], it will be considered here in the context of the Earth's climate system.

Forced by incoming visible solar radiation and damped by infrared radiation emitted to space with a distribution of net forcing segregated by latitude, the climate system is approximately in a NESS. While this approximation is violated by non-steady forcings such as solar variability, seasons and Milankovitch cycles in the Earth's orbit, intermittent volcanic eruptions, and anthropogenic greenhouse forcing, much remains to be learned about the steady-state climate and here we will ignore non-steady forcings. The climate system is known to exhibit many self-organized, irregular, spatio-temporal patterns, typically referred to as oscillations. These patterns include the El-Niño Southern Oscillation (ENSO) [9], the Madden-Julien Oscillation (MJO) [10], the Pacific Decadal Oscillation (PDO) [11], and the Atlantic Multidecadal Oscillation (AMO) [12]. It should be emphasized that these "oscillations" are not single-frequency constant amplitude sinusoidal fluctuations or necessarily wavelike phenomena: their frequency distributions and amplitude variations are broad and important, but they are narrow enough that each one is an empirically recognized coherent spatio-temporal pattern of natural variability. ENSO and the MJO are emergent phenomena that result from a complex "organization" of tropical convection, ocean temperatures, and large-scale oceanic and atmospheric waves over the Indian Ocean [13, 14]. As such, they are unlikely to obey detailed balance or any particular time-reversal symmetry. Nonetheless, ENSO and the MJO are the dominant modes of equatorial interannual and tropical intraseasonal variability, respectively. So, ENSO and the MJO both emerge from a multiplicity of mechanisms, but dominate other variability in their region and timescale. Here we interpret these climate oscillations as fluctuations about the mean and the specific spatio-temporal character of the oscillations is seen as a physical-space manifestation of the probability currents in the phase space‡ of the climate system.

Climate oscillations are often characterized by "indices", empirically determined combinations of climate variables. It is most common to focus on a single index, however sometimes two indices are used to describe an oscillation and investigate the trajectories of the indices in the resulting two-dimensional phase space. Climate oscillations are then observed to have trajectories which exhibit phase space rotation. For example, ENSO is often described in terms of the NINO3 index, based on the spatially-averaged Sea Surface Temperature in the eastern tropical Pacific (90°W to 150°W and 5°S to 5°N), and the average depth of the 20°C isotherm over the same area, which is a measure of the volume of warm water in the tropical Pacific. The two-dimensional phase space of these indices clearly shows the rotation characteristic of fluctuations within NESS, as seen in Figure 1a (e.g. [9]). Similar phase space rotation is seen (Figure 1b) in a multivariate MJO index [15] based on spatial patterns of variability of outgoing longwave radiation anomalies. The proposed probability angular momentum provides a natural measure that quantifies the nonequilibrium phase space currents. It is readily computed from observations, and allows comparisons between models and observations and for model

‡ In the physics community, "phase space" is a term used for the space of x-p (coordinate and momentum). Significantly, these variables are even/odd under time reversal. In this paper, we use this term in the sense common in the community of dynamical systems and climate science. In the cases we consider here, there is no reason to regard the variables (e.g., temperature and volume, or two amplitudes of a principal component analysis) as having different symmetry under time reversal.

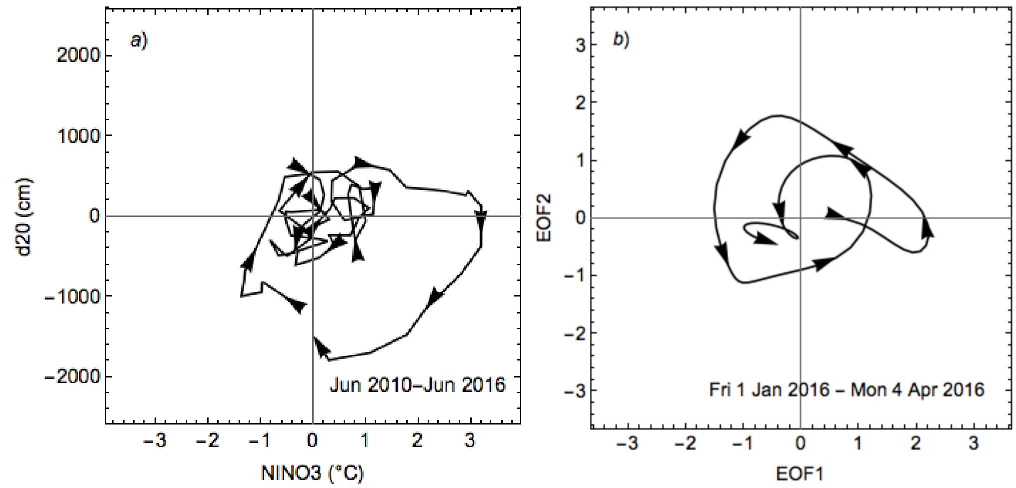

Figure 1. Phase space trajectories for a) ENSO and b) the MJO

intercomparisons.

One of the simplest classes of models that captures nonequilibrium steady-states, often used in the physics community, are Langevin models based on multivariate linear stochastic differential equations with additive Gaussian noise, known as linear inverse models in the community of climate science, e.g. [16]. They are generally assumed to have Gaussian noise (Linear Gaussian Model: LGM) and they have been successfully used to describe a variety of climate oscillations. The simplicity of these LGMs allows the properties of the nonequilibrium steady-state to be calculated analytically. We emphasize that although it is useful to analyze the probability angular momentum in the context of these simple LGMs, the quantity itself is quite general and captures the phase space rotation for any system regardless of the underlying dynamics.

We begin in section 2 with a review of the physics of nonequilibrium steady-states and probability currents for LGMs. Section 3 introduces the probability angular momentum - its average as well as its full distribution. In section 4 we calculate the probability angular momentum of two climate oscillations: ENSO and the MJO. We end with a summary and outlook.

2. Nonequilibrium Steady-states and Probability Currents

The physics of general nonequilibrium steady-states is extremely challenging. Some progress towards that goal is possible by considering systems described by LGMs. Of course, like most physical systems, the dynamics of the global climate is far from being linear and the stochastics are more complex than additive Gaussian noise. Nevertheless, if we focus on fluctuations near a stationary state, we may hope that these fluctuations are small and the essential properties of these fluctuations can be captured by a linearized theory. Similarly, we regard the use of a continuous time Markov process with additive noise as a reasonable first step towards more realistic descriptions of the complex system. Restricting our attention to *stable* stationary states, the LGM is completely specified by two matrices, one characterizing the deterministic relaxation into the stationary state ($\mathbf{A}$) and another describing the covariance of the noise ($\mathbf{D}$). Stability of the system requires the real parts of the eigenvalues of $\mathbf{A}$ to be negative, while $\mathbf{D}$, also known as the diffusion matrix, must be positive definite.

To be specific, we consider a system with an N-dimensional real state-vector $\vec{x} \in \mathbb{R}^N$, governed by the stochastic differential equation (SDE)

$$\frac{d\vec{x}}{dt} = \mathbf{A}\vec{x} + \mathbf{F}\vec{\xi}, \tag{1}$$

where $\mathbf{A}$, $\mathbf{F} \in \mathbb{R}^N \times \mathbb{R}^N$ are real $N \times N$ matrices, and $\vec{\xi}$ is a Gaussian distributed random variable with $\left\langle \vec{\xi} \right\rangle = 0$ and $\left\langle \vec{\xi}\vec{\xi}^T \right\rangle$ being the identity matrix (superscript T representing the transpose). Apart from the damping matrix $\mathbf{A}$, $\mathbf{F}$ here is related to the diffusion matrix by $\mathbf{D} \equiv \mathbf{F}\mathbf{F}^T/2$, which guarantees the positivity of $\mathbf{D}$. A Takagi factorization [17] can be used to determine $\mathbf{F}$ from $\mathbf{D}$ if only the latter is given. This LGM is equivalently described by the Fokker-Planck equation for its probability distribution function (pdf)

$p(\vec{x},t)$, which can be written in flux-gradient form as

$$\frac{\partial p}{\partial t} + \nabla \cdot (\vec{u}p) = 0, \tag{2}$$

where the phase space velocity is $\vec{u} = \mathbf{A}x - \mathbf{D}\nabla \ln p$. Focusing on the stationary§ pdf, $p^*(\vec{x})$, the solution is a Gaussian [18]

$$p^*(\vec{x}) = e^{-\vec{x}^T \mathbf{C}_0 \vec{x}/2} \Big/ \sqrt{\det \mathbf{C}_0}. \tag{3}$$

Here, $\mathbf{C}_0 = \left\langle \vec{x}\vec{x}^T \right\rangle^*$ is the covariance matrix in the stationary state and is related to $\mathbf{A}$ and $\mathbf{D}$ by the generalized fluctuation dissipation relation [18, 19] (Einstein relation):

$$\mathbf{A}\mathbf{C}_0 + \mathbf{C}_0\mathbf{A}^T + 2\mathbf{D} = 0. \tag{4}$$

From the Fokker-Planck equation (2), we see that $\nabla \cdot (\vec{u}^* p^*) = 0$ in a steady-state. There are two ways for this equation to be satified. One case is $\vec{u}^* \equiv 0$, in which case the system can be regarded as in thermodynamic equilibrium, as the dynamics satisfies detailed balance. The other case is $\vec{u}^* \neq 0$, in which case the system is in a NESS, with a nonzero, divergence-free, probability current $\vec{J}^* = \vec{u}^* p^*$. This current is related to the fundamental matrices through $\vec{u}^* = \mathbf{\Omega}\vec{x}$, $\mathbf{\Omega} = \mathbf{A} + \mathbf{D}\mathbf{C}_0^{-1}$, where $\mathbf{\Omega}$ is the phase space rotation frequency. It carries probability from one state to another in preferred directions. Being divergence free, the probability currents must form close loops, leading to the notion of rotations and "angular momenta" - the main focus of the next section.

§ To avoid possible confusion when the same symbol is used for both the general case and steady states, we will use the superscript * to denote quantities associated with the stationary state. In particular, $\langle \mathcal{O} \rangle^*$ below refers to the average in the steady state: $\int \mathcal{O}(\vec{x})\, p^*(\vec{x})\, d\vec{x}$.

3. Probability Angular Momentum, its Generalization and Distribution

Given that we expect rotations in phase space, we seek a quantitative measure of this behavior, a quantity which can be calculated from a time series $\vec{x}\,(t)$. In physical space, a standard measure of rotation is angular momentum. For a point particle with mass m, position $\vec{r}$, moving with velocity $\vec{v}$, its angular momentum is $\vec{L} = \vec{r} \times m\vec{v}$. Since we are interested in the angular momentum of a continuous distribution, we may consider the angular momentum of a rotating fluid in three dimensions with mass density $\rho(\vec{r}, t)$. From an Eulerian perspective, a fluid at position $\vec{r}$ has velocity $\vec{v}(\vec{r}, t)$ and its angular momentum *density* is $\vec{L}(\vec{r}, t) = \vec{r} \times \rho\vec{v}$. The total angular momentum of the fluid in the Eulerian perspective is then ‖

$$\vec{L}_E(t) = \int d\vec{r}\,\vec{L}(\vec{r}, t) = \int d\vec{r}\,\rho(\vec{r}, t)\,\vec{r} \times \vec{v}. \tag{5}$$

Alternatively, one can consider the fluid from a Lagrangian perspective, i.e., as a collection of fluid parcels. The trajectory of the i^{th} parcel is $\vec{X}_i(t; \vec{r}_0)$ where $\vec{r}_0$ is the parcel's initial position: $\vec{X}_i(0; \vec{r}_0) = \vec{r}_0$. Each parcel has mass m_i, the fluid density is $\rho(\vec{r}, t) = \sum_i m_i \delta(\vec{r} - \vec{X}_i)$, and a parcel's velocity is the Eulerian velocity at its location, $\vec{v}_i = \vec{v}(\vec{X}_i, t)$. A parcel then has angular momentum $\vec{X}_i \times m_i \vec{v}_i$ and the total fluid angular

‖ Note that we use the same letter for both the density of a quantity and the total, the former having an additional argument, $\vec{x}$. In particular, $\vec{L}\,(\vec{x}, t)$ is the angular momentum *density*, while $\vec{L}\,(t)$ is the total.

momentum in the Lagrangian perspective is

$$\vec{L}_L(t) = \sum_i \vec{X}_i \times m_i \vec{v}(\vec{X}_i), \tag{6}$$

$$= \int d\vec{r} \sum_i m_i \delta(\vec{r} - \vec{X}_i)\, \vec{r} \times \vec{v}(\vec{r}), \tag{7}$$

$$= \int d\vec{r}\, \rho(\vec{r}, t)\, \vec{r} \times \vec{v}(\vec{r}), \tag{8}$$

$$= \vec{L}_E(t), \tag{9}$$

which, of course, is the same as in the Eulerian perspective. From here, we propose the *probability* angular momentum in phase space as a straightforward analogue of this fluid angular momentum, by letting $\vec{r} \to \vec{x}$, $\vec{v} \to \vec{u}$ and $\rho(\vec{r}, t) \to p(\vec{x}, t)$. Since p is normalized to unity, the analogue of total mass is simply unity. Thus, the Eulerian fluid description carries over to the Fokker-Planck viewpoint of focusing on p in (2). Meanwhile, the Lagrangian fluid description is analogous to the SDE viewpoint of following individual trajectories in (1).

In an N dimensional phase space, the angular momentum is no longer a (pseudo-)vector but a (pseudo-)tensor, and instead of the vector $\vec{L}$, we represent it with an antisymmetric tensor L, $\mathsf{L}^T = -\mathsf{L}$. The α, β element of L, $L_{\alpha\beta}$, describes rotation in the two-dimensional α-β subspace. The *probability angular momentum density* at a point in phase space is now written in terms of the wedge product $\mathsf{L}(\vec{x}, t) = p(\vec{x}, t)\, [\vec{x} \wedge \vec{u}]$, or $L_{\alpha\beta}(\vec{x}, t) = p(\vec{x}, t)\, [x_\alpha u_\beta - x_\beta u_\alpha]$. Then, the total probability angular momentum at any time t, $\int d\vec{x} \mathsf{L}(\vec{x}, t)$, is just the average¶

$$\mathsf{L}(t) = \int d\vec{x}\, p(\vec{x}, t) \left[\vec{x} \wedge \frac{d\vec{x}}{dt} \right] = \left\langle \vec{x} \wedge \frac{d\vec{x}}{dt} \right\rangle, \tag{10}$$

¶ For readers who are concerned with the singular nature of $d\vec{x}/dt$ in SDE's, it is simplest to use an Ito formulation with discrete time steps ε. Then, this velocity is approximated by $[\vec{x}\,(t + \varepsilon) - \vec{x}\,(t)]/\varepsilon$. For observational data or climate models, such an approximation is necessary, as discussed below.

(with elements $L_{\alpha\beta}(t)$).

Turning to the steady state, we first note that $\mathbf{L}^* = \int \vec{x} \wedge \vec{J}^*$ in general, so that $\mathbf{L}^*$ is independent of the choice of origin of $\vec{x}$ (since $\vec{\nabla} \cdot \vec{J}^* = 0$, $\int \vec{J}^* = 0$). In the LGM (1), this total angular momentum takes a simple form:

$$\mathbf{L}^* \equiv \langle \vec{x} \wedge \mathbf{A}\vec{x} \rangle^* , \tag{11}$$

$$= \mathbf{C}_0 \mathbf{A}^T - \mathbf{A}\mathbf{C}_0, \tag{12}$$

$$= \; - 2\mathbf{\Omega}\mathbf{C}_0 \tag{13}$$

where we have used relation (4). This is the generalization of the familiar $\vec{L} = \mathbf{I}\vec{\omega}$ for rotating rigid bodies, as we note that the steady-state covariance $\mathbf{C}_0$ plays the role proportional to that of the inertia tensor $\mathbf{I}$. If and only if the system is in thermodynamic equilibrium or detailed balance then there is no rotation and $\mathbf{L}^* = \mathbf{\Omega} = 0$. Finally, Eqns. (4) and (12) reveal a remarkable relation, namely, $\mathbf{D}$ and $\mathbf{L}^*/2$ are the symmetric and antisymmetric parts of one matrix: $-\mathbf{A}\mathbf{C}_0$, *putting the probability angular momentum on the same footing as diffusion.* In this regard, it is crucial that the units of $\mathbf{L}^*$ are indeed those of diffusion, as noted above.

So far, our study of probability angular momenta has led us to equal time correlation functions (though the presence of $d\vec{x}/dt$ implies, rigorously, unequal times), just as $\vec{L}(t)$ for point masses involve $\vec{r}$ and $\vec{v}$ at one time, t. Though rarely considered in classical mechanics of point particles, there is a natural generalization of $\vec{L}(t)$ (given a known trajectory $\vec{r}(t)$), namely,

$$\vec{A}(t,t') \equiv m\vec{r}(t) \times \vec{r}(t') \tag{14}$$

Note that the magnitude $\left|\vec{A}\right|$ is the area of a parallelogram spanned by the two $\vec{r}$'s (related to the area in Kepler's second law, of course). Clearly, as $t' \to t$ from above, $\vec{L}$ is recovered, i.e., $\vec{L}(t) = d\vec{A}(t,t')/dt'\Big|_{t'=t}$. For the analog in statistical mechanics, we consider the two point correlation function at unequal times, $C_{\alpha\beta}(t,t') = \langle x_\alpha(t) x_\beta(t')\rangle$. To be precise, the definition

$$\langle x_\alpha(t) x_\beta(t')\rangle \equiv \int x_\alpha x'_\beta P(\vec{x},t;\vec{x}',t')\, d\vec{x}\, d\vec{x}' \tag{15}$$

requires the *joint* probability distribution for the system, $P(\vec{x},t;\vec{x}',t') = p(\vec{x},t)\, G(\vec{x}'-\vec{x},t'-t)$. Here, $G\left(\vec{\xi},\tau\right)$ is the time dependent solution to the Fokker-Planck equation (2), subjected to the initial condition $G\left(\vec{\xi},0\right) = \delta\left(\vec{\xi}\right)$. The analog of $\vec{A}(t,t')$, i.e., the generalization of $L_{\alpha\beta}(t)$, is just the antisymmetric combination

$$\tilde{C}_{\alpha\beta}(t,t') \equiv \langle x_\alpha(t) x_\beta(t')\rangle - \langle x_\alpha(t) x_\beta(t')\rangle \tag{16}$$

with $L_{\alpha\beta}(t) = d\tilde{C}_{\alpha\beta}(t,t')/dt'\Big|_{t'=t}$. By construction, $\tilde{C}$ is odd when the t's are exchanged: $\tilde{C}_{\alpha\beta}(t,t') = -\tilde{C}_{\alpha\beta}(t',t)$, a property related to violation of time reversal symmetry. Of course, in the steady state $\tilde{C}^*$ is stationary, so it depends only on the difference $\Delta t \equiv t - t'$ and is odd in Δt. This formulation parallels that of relative dispersion in compressible fluids [20, 21, 22].

Naturally, computing these quantities is difficult in general. However, as the LGM is completely specified by the matrices $\mathbf{A}$ and $\mathbf{D}$, much can be found analytically by exploiting the LGM once it is found, especially in the steady state. In particular, the time-lagged covariance $\mathbf{C}_\tau$ is related simply to $\mathbf{C}_0$:

$$\mathbf{C}_\tau = \left\langle \vec{x}(t+\tau)\vec{x}^T(t)\right\rangle^* = e^{\mathbf{A}\tau}\mathbf{C}_0, \tag{17}$$

which is independent of t, due to time translational invariance of a steady-state. Instead of $\mathbf{A}$ and $\mathbf{D}$, the LGM can be specified alternatively by these steady-state covariance matrices. The advantage is that this representation is useful for constructing empirical models from data as discussed below. Further, note that $d\mathbf{C}_\tau/d\tau|_{\tau=0} = \mathbf{A}\mathbf{C}_0$ contains the full information contained in $\mathbf{D}$ and $\mathbf{L}^*$. Thus, a third representation of the SDE (1) is to specify the LGM in terms of $\mathbf{C}_0$, $\mathbf{D}$ and $\mathbf{L}^*$. The advantage of this representation is that all possible systems can be grouped into families with the same $\mathbf{C}_0$ and $\mathbf{D}$ but different $\mathbf{L}^*$. To emphasize, only one member of the family (the one with $\mathbf{L} = 0$) is in thermodynamic equilibrium. All other members represent nonequilibrium systems with the same pdf and diffusion, but different phase space rotations and thus, different spatio-temporal patterns of variability (and different predictability).

Turning to data, we note that observations of the climate system necessarily follow the trajectory of the single climate system we live in. By assuming that the climate system is in a NESS one can use ergodicity to compute the properties of the steady-state. Increasingly, however, climate models are being run with ensembles of trajectories, which allows us to calculate properties of the pdf in a time-evolving climate (e.g., [23]). Here, however, we restrict ourselves to steady-state climates. In all cases, the trajectories are recorded at integer multiples of a finite time step, τ. Thus, instead of the true velocity of the trajectory, we have the finite difference approximation to the velocity, $d\vec{x}/dt \approx (\vec{x}(t+\tau) - \vec{x}(t))/\tau$, and the total probability angular momentum in the NESS, $\mathbf{L}^*$, is approximated by

$$\mathbf{L}^*_\tau \equiv \langle \vec{x}(0) \wedge \vec{x}(\tau)/\tau \rangle^* \,. \tag{18}$$

Under the assumption of ergodicity $\mathbf{C}_\tau$ is calculated from a *time average* over a single trajectory rather than a phase space average. In this manner, $\mathbf{L}^*_\tau$ computed from typical climate observations and model runs provides a quantification of the rotational properties of climate oscillations.

As a trajectory, $\vec{x}(t)$, moves in phase space (through the steady-state pdf), the associated finite-time probability angular momentum $\vec{x}(t) \wedge \vec{x}(t+\tau)/\tau$ samples a distribution of values. Considerable information is carried in such a distribution (e.g., the variance associated with the average $\mathbf{L}_\tau^*$). To present a study of this distribution, let us focus on a two dimensional phase space for simplicity. Then, the time series of a single (independent) quantity is associated with wedge product

$$\mathcal{L}_\tau(t) \equiv [x_1(t)x_2(t+\tau) - x_2(t)x_1(t+\tau)]/\tau$$

which can be used to compile a histogram $H(\mathcal{L}_\tau)$. Note that, in general, $\mathcal{L}_\tau(t)$ will appear with both signs and the support of H is over the entire line $(-\infty, \infty)$. Normalizing H provides us with a pdf, which can be compared to the theoretical expression

$$f(\mathcal{L}_\tau) = \int \delta\left(\mathcal{L}_\tau - \frac{x_1 x_2' - x_2 x_1'}{\tau}\right) P^*(\vec{x}, 0; \vec{x}', \tau)\, d\vec{x}\, d\vec{x}'$$

where $P^*(\vec{x}, 0; \vec{x}', \tau) = p^*(\vec{x})\, G(\vec{x}' - \vec{x}, \tau)$ is the joint probability in NESS. In the LGM, G is also Gaussian, like p^*, so that the Fourier transform of f

$$\hat{f}(\phi) = \int f(\mathcal{L}_\tau) e^{-i\mathcal{L}_\tau \phi} d\mathcal{L}_\tau \tag{19}$$

involves only Gaussian integrals and can be computed exactly. Deferring technical details to another publication, let us summarize the main results here.

- $1/\hat{f}(\phi)$ is the square root of a determinant (of the 4×4 matrix appearing in the Gaussian), which is a quartic polynomial in ϕ.
- The parameters of this polynomial come from the defining matrices of the LGM: either the pair ($\mathbf{A}$,$\mathbf{D}$), or the set ($\mathbf{C}_0$,$\mathbf{D}$,$\mathbf{L}^*$). Note that, in our case, $\mathbf{L}^*$ has only one independent element. We denote its off-diagonal $1, 2$ element by ℓ.
- As expected, this polynomial is symmetric in $i\phi$ if $\ell = 0$, so that the distribution f

is also symmetric in $\mathcal{L}_\tau$, leading to $\langle \mathcal{L}_\tau \rangle \equiv 0$. Note that f is not δ distributed and has a finite variance, denoted by σ_0^2. The physics is clear: A trajectory for a system in an equilibrium steady state is just as likely to rotate one way as the other, with no violation of time reversal symmetry.

- The singularities of $\hat{f}(\phi)$, located at the roots of the quartic, are branch points, which lie on both sides of the real axis.
- The branch points nearest the real axis dictate the large $\mathcal{L}_\tau$ asymptotic exponential decay of $f(\mathcal{L}_\tau)$.
- Of course, $\langle \mathcal{L}_\tau \rangle = i d\hat{f}/d\phi\big|_{\phi=0}$, which is just ℓ. Further, $\left\langle {\mathcal{L}_\tau}^2 \right\rangle = -d^2\hat{f}/d\phi^2\big|_{\phi=0} = \sigma_0^2 + 2\ell^2$. As a result, we arrive at a simple expression

$$\sigma_\ell^2 \equiv \left\langle {\mathcal{L}_\tau}^2 \right\rangle - \langle \mathcal{L}_\tau \rangle^2 = \sigma_0^2 + \ell^2$$

for the variance of the distribution f for systems in NESS. This leads to an important ratio

$$\frac{\langle \mathcal{L}_\tau \rangle}{\sigma_{\mathcal{L}}} = \sqrt{\frac{\ell^2}{\sigma_0^2 + \ell^2}} \tag{20}$$

which implies the following caution. If a trajectory with finite time steps in phase space is used to find averages and standard deviations of probability angular momenta, and if a NESS system is well described by an LGM, then $\langle \mathcal{L}_\tau \rangle$ can never exceed $\sigma_{\mathcal{L}}$. Thus, we must examine the statistics of the full pdf in order to come to a meaningful conclusion on whether a nonzero $\langle \mathcal{L}_\tau \rangle$ is significant or not. In contrast, stochastic processes with prominent rotational aspects (e.g., noisy limit cycles) are not subjected to the limitations shown here [24]. A comprehensive discussion is beyond the scope of this work and will be presented elsewhere.

Within the context of the LGM, we presented a complete analytic description of various aspects of the probability angular momenta. For systems that display prominent rotations, there is little need to identify them as NESS. But, there are many cases where the trajectories in phase space display subtle rotations, hidden behind a substantial amount of noise. Then a non-vanishing $\langle \mathcal{L}_\tau \rangle$ and a significantly asymmetric $f(\mathcal{L}_\tau)$ are clear signals that the system cannot be regarded as "in equilibrium." Of course, a natural question is how we interpret the sign of $\langle \mathcal{L}_\tau \rangle$. Our proposal is that, especially in cases of "subtle displays" of this rotation, one of the variables is the "driver" with the other being the "follower," much like the increase of prey populations "drives" the increase in the numbers of predators. In physical systems, the sign of $\langle \mathcal{L}_\tau \rangle$ may point us to more tractable underlying causes of this "driver-follower" behavior, a key characteristic of NESS. In the next section, we will apply these findings to the climate system, focusing on just two specific phenomena.

4. Example Climate Oscillations: ENSO and MJO

Climate oscillations are preferred spatio-temporal patterns of natural variability of the climate system. Each oscillation has its own timescale and has a large projection onto different, relatively small subspaces of the massively high-dimensional phase space of the entire climate system. Climate oscillations are studied with climate indices: functions of subsets of climate variables, filtered to the specific spatio-temporal scales of the pattern, and empirically developed to capture the dominant features of a climate oscillation with one, two, or a few scalar quantities. Oscillations often have many competing indices, each of which highlights a different aspect of the complex pattern.

The El-Niño Southern Oscillation (ENSO) has its largest projection in the tropical Pacific region, dominating ocean temperatures, the location of convection, and the atmospheric Walker circulation. The timescale of individual ENSO events is roughly 9 months and the time between events is on the order of 2 to 7 years. We will describe ENSO in terms of the NINO3 index (NINO3) and the depth of the 20°C isotherm in the tropical Pacific (d20). ENSO index data is publicly available from a number of sources. Here we use the data from the KNMI Climate Explorer ([25]). The data consists of monthly averages of observations and extends from 1960 to 2016.

The Madden-Julien Oscillation (MJO) is an eastward moving pattern that has its largest projection on tropical rainfall, convection, and outgoing longwave radiation. Its timescale is weeks to months. We will describe the MJO in terms of the so-called Original MJO Index (OMI) which is a two-dimensional index representing the principal components of the first two empirical orthogonal functions (EOFs) of filtered outgoing longwave radiation. Like ENSO data, MJO data is publicly available from a number of sources. Here we use the data from NOAA's Earth System Radiation Lab. The data is daily and extends from 1/1/1979 - 4/26/2016.

The units of phase space for climate indices can be unintuitive. The units of probability angular momentum are the same as the units of diffusion, length^2/time. The two ENSO indices, however, have different units of "length". NINO3 is a temperature anomaly and has units of °C, while d20 is a depth anomaly and has units of cm. As a result, in the NINO3-d20 phase space with monthly data the probability angular momentum has units of °C cm/month. Sometimes one uses indices that have been scaled by their standard deviation resulting in indicies which are unitless. Then, the probability angular momentum would have units of 1/time.

Climate oscillations can be modeled by linear Gaussian models of the form (1) through a process called linear inverse modeling. A multivariate time series is used to construct the steady-state covariance $\mathbf{C}_0$ and the time-lagged covariance $\mathbf{C}_\tau$. The time lag is empirically chosen to capture the timescale of the climate oscillation of interest.

Then Eq. (17) is used to compute $\mathbf{A}$ and Eq. (4) determines $\mathbf{D}$. There is no guarantee that this procedure will result in a stable SDE, but it often works surprisingly well, e.g., [26, 27, 28, 29]. Here we use linear inverse modeling to construct two-dimensional linear Gaussian models for ENSO and the MJO from the time series of their indices.

The pdf $f(\mathcal{L}_\tau)$ can be calculated directly from the time series as well as theoretically from the linear Gaussian models. The pdf's are strongly asymmetric and have exponential tails. The asymmetry leads to the total probability angular momentum of the steady-state, $\langle\mathcal{L}_\tau\rangle$, being nonzero. The two methods of generating a pdf agree surprisingly well. The linear inverse modeling procedure by constructions produces a linear Gaussian model that has same total $\langle\mathcal{L}_\tau\rangle$ as the data. The strong agreement of the full pdf suggests that measurements of $f(\mathcal{L}_\tau)$ are robust and they can be used to verify models (both LGMs and more complex climate models) and for model intercomparison. The two examples shown here were minimal, involving only two degrees of freedom, but the approach and equations presented here can be applied to more detailed systems as well, keeping (20) as a guide for how many probability angular momenta about various axes are reliable.

5. Summary and Outlook

In this work, we focus on a principal characteristic of nonequilibrium steady-states, namely, persistent probability currents. We discuss their observable consequences in the context of climate science, illustrating with two specific "oscillatory" phenomena: ENSO and the MJO. To quantify these manifestations of probability currents, we propose a new quantity: *probability angular momentum density* in phase space: $\mathbf{L}(\vec{x},t) \equiv \vec{x} \wedge \vec{u}p(\vec{x},t)$. Focusing on the steady state properties, we investigated various facets associated with this concept beyond the simple time-independent average $\mathbf{L}^* = \langle \vec{x} \wedge \vec{u} \rangle^*$, such as (i)

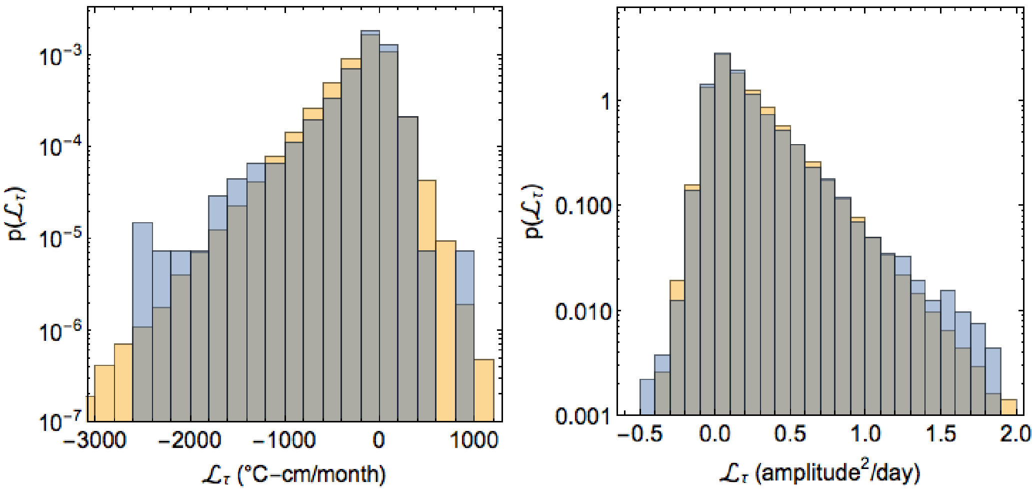

Figure 2. Pdf's of the finite-time probability angular momentum for observations and a linear Gaussian model to the observations. a) ENSO using monthly data, and b) MJO using daily data. Gray indicates regions where the two pdfs overlap, yellow indicates regions where the model pdf is larger than the observation pdf, and blue indicates regions where the observation pdf is larger than the model pdf.

its relationship to diffusion matrix $\mathbf{D}$, (ii) its generalization to finite time averages $\mathbf{L}^*_\tau \equiv \langle \vec{x}(0) \wedge \vec{x}(\tau)/\tau \rangle^*$, (iii) its intimate connection with two point correlations at unequal times, and (iv) its probability distribution function $f(\mathcal{L}_\tau)$. Exact, analytic expressions for all quantities are obtained within the linear Gaussian model (LGM) and the case for systems with two-dimensional phase space is explicitly shown. To apply this analysis to data from observations or computer models, we construct the time-lagged covariance matrix from the time series $\vec{x}(t)$ and invoke the ergodicity assumption, so that we can compare time averages from data with ensemble averages from the theory.

Proposing a new quantity - probability angular momentum - to characterize fluctuations in NESS systems, this study provided new insights on its role in revealing the underlying time reversal-violating dynamics. Naturally, many new questions arise. One line of questions follows the applications to the climate system. Needless to say, there are many, many more climate phenomena to which we can apply this type of analysis in addition to those already mentioned–the North Atlantic Oscillation, Southern and Northern Annular Modes, variability of the Oceanic Meridional Overturning Circulation, etc.–as well as other physical systems. The accuracy of models of these phenomena can usefully be constrained by evaluating the probability angular momenta of the model versus those of observations. The other line of pursuit is in the realm of theory. Many issues related to the probability angular momentum, within the context of the LGM, remain to be explored further. A prime example is the time-lagged correlation $\langle \vec{x}(0) \wedge \vec{x}(\tau) \rangle^*$, for general τ, and the associated pdfs. The few preliminary investigations for specific systems [7, 8] should be extended to a comprehensive study, valid for all LGMs in arbitrary dimensional phase space. What are the key characteristics of the LGM that can lead us to predict which are the "driving variables" and which are the "followers?" Beyond these questions associated with the probability currents, loops, angular momenta, and rotations in phase space, we should consider their implications in a wider context. Can such considerations lead us to a quantity, or quantities, beyond the twin pillars of equilibrium statistical mechanics: energy and entropy? From these steps, we may find hints towards how to formulate a framework for preferred fluctuations in non-equilibrium statistical mechanics and the climate system.

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